Journal of Vibration and Control
http://jvc.sagepub.com

Calculus Model for a Rolling Guided Missile

R.M. Botez, V. Chelaru, P. Parvu and C. Gheorghe
Journal of Vibration and Control 2001; 7; 863
DOI: 10.1177/107754630100700606

Calculus Model for a Rolling Guided Missile

R. M. BOTEZ
École de technologie supérieure, Département de génie de la production automatisée, 1100 Notre Dame Ouest, Montréal, Québec, Canada, H3C 1K3

V. CHELARU
Uzina Electromecanică, Ploieşti, Romania

P. PARVU
Universitatea Politehnica Bucureşti, Facultatea de Inginerie Aerospaţială, Bucureşti, Romania

C. GHEORGHE
Uzina Electromecanică, Ploieşti, Romania

(Received 7 September 2000; accepted 14 February 2001)

Abstract: The paper presents the relations that form the basis for the calculus models needed to study rolling guided missiles. The equations of the general motion are written within the Resal trihedral in a way that allows, after linearization, one to draw up a structural scheme of the controlled object, a scheme that will be used in the guided missile system analysis. Opposite to the flying machines with stabilized roll where the analysis is done after decoupling the channels, for the missile with rotation, the analysis of the scheme is done simultaneously on both channels, as terms used are complex terms. Furthermore, there will be presented some characteristics of this type of missile by analyzing a model close to the one of an antichariot missile with gasodynamic control.

Key Words: Guidance, control, missiles, gasodynamics, flight quality, aerodynamics

NOMENCLATURE

The basic notations are according to standards presented by Niţă et al. (1999a, 1999b, 1999c).

- $Ox^*y^*z^*$ = the Resal trihedral
- u^*, v^*, w^* = velocity $\bar{V}$ components within the Resal trihedral
- p^*, q^*, r^* = angular velocity $\bar{\Omega}^*$ components within the Resal trihedral
- $\dot{\Phi} = \omega_x$ = the angular velocity of the mobile trihedral (attached to the missile) relative to the Resal trihedral
- A, B, C = axial moments of inertia.

In the following, we shall write the equations of the general motion, this time within the Resal trihedral.

Journal of Vibration and Control, 7: 863-889, 2001

1. INTRODUCTION

In modern warfare, the rolling missiles have a significant impact in fighting the armored vehicles as well as in close-range air defense. In the air defense role, due to the high angular velocity of the line-of-sight to target, the missile is directly self-guided, using the proportional navigation method. This method does not ensure a direct hit but it provides a close enough encounter to trigger the proximity fuse and destroy the target, which, as an aircraft, has no armor. In the anti-armor role, the missiles are either remote guided by wire or indirectly self-guided by means of a laser or radio beam. In this case, a direct hit is needed on the heavy armored target and the angular velocity of the line-of-sight is low, so these kinds of missiles use the "3 points guidance" methods. The wire remote-guided missiles use the angular error control, while the indirect-guided (either laser or radio beam) missiles use the linear error control.

If the flight speed regime is subsonic, the aerodynamic control surfaces have little efficiency and the missiles use gasodynamic controls. In the case of supersonic missiles, either laser- or radio-guided aerodynamic controls can be used. The advantage of the rolling missiles is, first of all, the annulment of the perturbation momentum due to aerodynamic or gasodynamic asymmetries. Another advantage is that, due to the roll motion, the command can be realized by one single channel, which means either a single pair of rudders or of mobile nozzles is used, thus reducing the size of the control system and the necessary energy resource. This paper presents the relations that form the basis for the calculus models needed to study these kinds of missiles.

2. GENERAL MOTION EQUATIONS

Unlike the case of classical guided missiles having a roll-stabilized motion, in the case of the rolling missiles, most authors such as Carpantier (1989a, 1989b, 1989c) and Kuzovkov (1976) use a mobile semilinked trihedral named "Resal," characterized by the fact that it does not participate in the roll motion of the missile (Figure 1). The advantage of this trihedral consists in the fact that the isolation of gyroscopic coupling terms brings the equations of motion to a form that is very close to those of the roll-stabilized missile, allowing the use of common study methods, at least in respect to their linear form.

In the following, we shall write the equations of the general motion within the Resal trihedral in a way that will allow us (after linearization) to draw up a structural scheme of the controlled object, a scheme that will be eventually developed and used in the guided missile system analysis.

To reformulate the general equations, we shall start by observing that the elements of the motion within the mobile trihedral attached to the missile ($Oxyz$) can be obtained from those within the Resal trihedral ($Ox^*y^*z^*$), by means of the following rotation matrix:

$$\mathbf{A}_\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix}. \tag{1}$$

At the beginning, we consider the dynamic motion equations written within the mobile trihedral, attached to the missile:

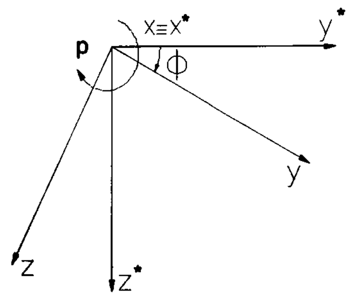

Figure 1. The Resal trihedral and the mobile (attached to the missile) trihedral.

$$
\begin{aligned}
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} &= \frac{1}{m}\left\{ \begin{bmatrix} X^A \\ Y^A \\ Z^A \end{bmatrix} + \begin{bmatrix} X^T \\ Y^T \\ Z^T \end{bmatrix} \right\} + \mathbf{A}_p \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} - \mathbf{A}_\Omega \begin{bmatrix} u \\ v \\ w \end{bmatrix}; \\
\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} &= \mathbf{J}^{-1}\left\{ \begin{bmatrix} L^A \\ M^A \\ N^A \end{bmatrix} + \begin{bmatrix} L^T \\ M^T \\ N^T \end{bmatrix} \right\} - \mathbf{J}^{-1}\mathbf{A}_\Omega \mathbf{J} \begin{bmatrix} p \\ q \\ r \end{bmatrix},
\end{aligned} \tag{2}
$$

where $\mathbf{A}_\Omega$ is the antisymmetric matrix associated to the angular velocity vector $\bar{\Omega}$ and $\mathbf{J}$ is the matrix of the moments of inertia for the axial-symmetric configurations case.

In the following, we shall try to express the main quantities of these equations by means of equivalent quantities within the semilinked trihedral. In this case, the velocity $\bar{V}$ components within the mobile, missile-linked trihedral become

$$
\begin{bmatrix} u & v & w \end{bmatrix}^T = \mathbf{A}_\Phi \begin{bmatrix} u^* & v^* & w^* \end{bmatrix}^T, \tag{3}
$$

and their derivatives are

$$
\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} \dot{u}^* \\ \dot{v}^* \\ \dot{w}^* \end{bmatrix} + \omega_x \frac{\partial \mathbf{A}_\Phi}{\partial \Phi} \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} \dot{u}^* \\ \dot{v}^* + \omega_x w^* \\ \dot{w}^* - \omega_x v^* \end{bmatrix}. \tag{4}
$$

Considering that

$$
\bar{\Omega} = \bar{\Omega}^* + \bar{\omega}_x, \tag{5}
$$

the angular velocity components within the "missile" trihedral will be expressed as follows:

$$\begin{bmatrix} p & q & r \end{bmatrix}^T = \mathbf{A}_\Phi \begin{bmatrix} \omega_x + p^* & q^* & r^* \end{bmatrix}^T, \tag{6}$$

and after derivation they become

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} \dot{\omega}_x + \dot{p}^* \\ \dot{q}^* \\ \dot{r}^* \end{bmatrix} + \omega_x \frac{\partial \mathbf{A}_\Phi}{\partial \Phi} \begin{bmatrix} \omega_x + p^* \\ q^* \\ r^* \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} \dot{\omega}_x + \dot{p}^* \\ \dot{q}^* + \omega_x r^* \\ \dot{r}^* - \omega_x q^* \end{bmatrix}. \tag{7}$$

On the other hand, the torsor components containing aerodynamic, gasodynamic, and mass terms will be expressed as follows:

$$\begin{aligned} \begin{bmatrix} X & Y & Z \end{bmatrix}^T &= \mathbf{A}_\Phi \begin{bmatrix} X^* & Y^* & Z^* \end{bmatrix}^T, \\ \begin{bmatrix} L & M & N \end{bmatrix}^T &= \mathbf{A}_\Phi \begin{bmatrix} L^* & M^* & N^* \end{bmatrix}^T \end{aligned} \tag{8}$$

and the antisymmetric matrix, attached to the angular velocity vector $\bar{\Omega}$, becomes

$$\mathbf{A}_\Omega = \mathbf{A}_\Phi \mathbf{A}_\Omega^{**} \mathbf{A}_\Phi^T, \tag{9}$$

where

$$\mathbf{A}_\Phi^{**} \begin{bmatrix} 0 & -r^* & q^* \\ r^* & 0 & -p^* - \omega_x \\ -q^* & p^* + \omega_x & 0 \end{bmatrix} = \mathbf{A}_\Omega^* + \omega_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{10}$$

Introducing these relations into equations (2), we obtain

$$\mathbf{A}_\Phi \begin{bmatrix} \dot{u}^* \\ \dot{v}^* + \omega_x w^* \\ \dot{w}^* - \omega_x v^* \end{bmatrix} = \frac{1}{m} \mathbf{A}_\Phi \begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix} - \mathbf{A}_\Phi \mathbf{A}_\Omega^{**} \mathbf{A}_\Phi^T \mathbf{A}_\Phi \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix}, \tag{11}$$

respectively.

$$\mathbf{J}\mathbf{A}_\Phi \begin{bmatrix} (\dot{p}^* + \dot{\omega}_x) \\ (\dot{q}^* + \omega_x r^*) \\ (\dot{r}^* - \omega_x q^*) \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} L^* \\ M^* \\ N^* \end{bmatrix} - \mathbf{A}_\Phi \mathbf{A}_\Omega^{**} \mathbf{A}_\Phi^T \mathbf{J} \mathbf{A}_\Phi \begin{bmatrix} (p^* + \omega_x) \\ q^* \\ r^* \end{bmatrix}. \tag{12}$$

Since the rolling missiles have axial-symmetric configurations, the cross moments of inertia are equal ($B = C$), and we can write

$$\mathbf{A}_\Phi^T \mathbf{J} \mathbf{A}_\Phi = \mathbf{J}. \tag{13}$$

In this case, multiplying at left the equations (12) and (13) by the inverse of the rotation matrix $\mathbf{A}_\Phi$, we obtain

$$\begin{aligned}
\begin{bmatrix} \dot{u}^* \\ \dot{v}^* \\ \dot{w}^* \end{bmatrix} &= \frac{1}{m}\begin{bmatrix} X^* \\ Y^* \\ Z^* \end{bmatrix} - \mathbf{A}_\Omega^* \begin{bmatrix} u^* \\ v^* \\ w^* \end{bmatrix}; \\
\begin{bmatrix} A\dot{p}^* \\ B\dot{q}^* \\ C\dot{r}^* \end{bmatrix} &= \begin{bmatrix} L^* \\ M^* \\ N^* \end{bmatrix} - \mathbf{A}_\Omega^* \begin{bmatrix} Ap^* \\ Bq^* \\ Cr^* \end{bmatrix} - \mathbf{A} \begin{bmatrix} \dot{\omega}_x \\ r^*\omega_x \\ -q^*\omega_x \end{bmatrix},
\end{aligned} \tag{14}$$

where the antisymmetric matrix associated with the angular velocity vector $\bar{\Omega}^*$ is

$$\mathbf{A}_\Omega^* = \begin{bmatrix} 0 & -r^* & q^* \\ r^* & 0 & -p^* \\ -q^* & p^* & 0 \end{bmatrix}. \tag{15}$$

Comparing these equations with equations (2), obtained by projection along the mobile missile-linked trihedral, we notice that inside the equations of moments a gyroscopic coupling term has been separated, which couples the longitudinal channels and is caused by the roll motion of the missile. Furthermore, to bring these equations to a more convenient form, the first line of matrix equation (6) leads to $p = p^* + \omega_x$, and the Euler dynamic equations are now expressed in the scalar form:

$$\dot{p} = \frac{L^*}{A}; \quad \dot{q}^* = \frac{M^*}{B} + r^*p^* - \frac{A}{B}r^*p; \quad \dot{r}^* = \frac{N^*}{C} - p^*q^* + \frac{A}{C}q^*p. \tag{16}$$

Regarding the cinematic equations, expressed with elements from the semilinked trihedral (Resal), they will be obtained from those written within the mobile missile-linked trihedral:

$$\begin{aligned}
\dot{x}_p &= u\cos\Theta\cos\Psi + v(\sin\Phi\sin\Theta\cos\Psi - \cos\Phi\sin\Psi) \\
&+ w(\cos\Phi\sin\Theta\cos\Psi + \sin\Phi\sin\Psi) \\
\dot{y}_p &= -u\cos\Theta\sin\Psi - v(\sin\Phi\sin\Theta\sin\Psi + \cos\Phi\cos\Psi) \\
&- w(\cos\Phi\sin\Theta\sin\Psi - \sin\Phi\cos\Psi) \\
\dot{z}_p &= u\sin\Theta - v\sin\Phi\cos\Theta - w\cos\Phi\cos\Theta,
\end{aligned} \tag{17}$$

respectively.

$$\begin{aligned}
\dot{\Phi} &= p + q\,\mathrm{tg}\,\Theta\sin\Phi + r\,\mathrm{tg}\,\Theta\cos\Phi; \\
\dot{\Theta} &= q\cos\Phi - r\sin\Phi; \\
\dot{\Psi} &= q\sin\Phi\sec\Theta + r\cos\Phi\sec\Theta
\end{aligned} \tag{18}$$

for the particular case of zero bank angle ($\Phi = 0$). In this way, from the Euler equations (18), we can obtain

$$\dot{\Theta} = q^*; \quad \dot{\Psi} = r^*\sec\Theta, \tag{19}$$

and the bank angular velocity definition relation becomes a link relation between the semilinked components:

$$p^* = -r^* \operatorname{tg}\Theta. \tag{20}$$

Finally, the other cinematic equations (17), which express the link between the velocity components inside the "earth" trihedral and inside the semilinked trihedral become

$$\begin{aligned}
\dot{x}_p &= u^* \cos\Theta\cos\Psi - v^* \sin\Psi + w^* \sin\Theta\cos\Psi \\
\dot{y}_p &= -u^* \cos\Theta\sin\Psi - v^* \cos\Psi - w^* \sin\Theta\sin\Psi; \\
\dot{z}_p &= u^* \sin\Theta - w^* \cos\Theta.
\end{aligned} \tag{21}$$

3. THE LINEAR FORM EQUATIONS OF THE RAPID MOTION AROUND THE CENTER OF MASS

Considering the basic motion of the rolling missiles as being a symmetric translation, and using the semilinked (Resal) trihedral, we can group and analyze together the equations of the rapid motion in the vertical and lateral planes. In this respect, we assume that the elements of the slow motion and of the roll motion are "frozen" at their equilibrium values, and the perturbation induced inside the system develops itself only by rapid motion around the mass center, which, as it will be subsequently shown, cannot be separately analyzed on each channel, because it is in fact a unique spiral-type evolution.

The analysis of this motion starts with the symmetrization of the force equations along the y^*, z^* axis of the Resal trihedral. To do this, we consider the fact that, due to the roll motion, the perturbation forces caused by aerodynamic and gasodynamic asymmetries are annihilated, the only perturbations remaining being caused by the gravitational acceleration (which affects also the lateral plane by the Magnus terms) and by the wind.[1] Even more, the roll motion introduces a series of Magnus coupling terms in the aerodynamic coefficients, on both channels. Considering all these observations, and linearizing the first equations (14), we obtain

$$\begin{aligned}
\Delta\dot{v}^* &= \frac{F_o}{m}C_{y\beta}\Delta\beta + \left(\frac{F_o l}{mV}C_{yr} - u^*\right)\Delta r^* + \frac{F_o l}{mV}C_{y\dot\beta}\Delta\dot\beta + \frac{F_o l}{mV}C_{yp\alpha}p\Delta\alpha \\
&+ \frac{F_o l^2}{mV^2}C_{yp\alpha}p\Delta q^* + \frac{F_o}{m}C_{y\delta n}\,\Delta\delta_n^* - \frac{F_o}{m}C_{y\beta}\Delta\beta_W + \Delta Y_p^*/m \\
\Delta\dot{w}^* &= \frac{F_o}{m}C_{z\alpha}\Delta\alpha + \left(\frac{F_o l}{mV}C_{zq} + u^*\right)\Delta q^* + \frac{F_o l}{mV}C_{z\dot\alpha}\Delta\dot\alpha + \frac{F_o l}{mV}C_{zp\beta}p\Delta\beta \\
&+ \frac{F_o l^2}{mV^2}C_{zpr}\,p\Delta r + \frac{T}{m}\Delta\delta_m^* - \frac{F_o}{m}C_{z\alpha}\Delta\alpha_W + \Delta Z_p^*/m,
\end{aligned} \tag{22}$$

where the angles of incidence are defined by velocity components along the Resal trihedral:

$$\alpha = \operatorname{arctg}(w^*/u^*); \quad \beta = -\operatorname{arctg}(v^*/u^*), \tag{23}$$

and the aerodynamic coefficients are also considered related to the semilinked trihedral:

$$C_y = Y^*/F_0; \quad C_z = Z^*/F_0; \quad C_m = M^*/H_0; \quad C_n = N^*/H_0.$$

To bring these equations to a form similar to those of the roll-stabilized missile, we must express the derivatives of the components of velocity in the Resal trihedral, in terms of angles of incidence derivatives. In this case, within the frame of the basic motion conditions ($\beta = 0$), if we also consider the longitudinal angle of incidence (α) as being small enough, by derivation of equations (23), we obtain the approximately symmetric form

$$\Delta\dot{v}^* = -u^*\Delta\dot{\beta}; \quad \Delta\dot{w}^* \cong u^*\Delta\dot{\alpha}, \tag{24}$$

and equations (22) can be written

$$\begin{aligned}
\Delta\dot{\alpha} &= \frac{F_o}{mu^*}C_{zu}\,\Delta\alpha + \left(\frac{F_o l}{mVu^*}C_{zq} + 1\right)\Delta q^* + \frac{F_o l}{mVu^*}C_{z\dot{\alpha}}\Delta\dot{\alpha} + \frac{F_o l}{mVu^*}C_{zp\beta}p\Delta\beta \\
&+ \frac{F_o l^2}{mV^2u^*}C_{zpr}\,p\Delta r^* + \frac{F_o}{mu^*}C_{z\delta m}\,\Delta\delta_m^* - \frac{F_o}{mu^*}C_{z\alpha}\Delta\alpha_W + \frac{1}{mu^*}\Delta Z_p^* \\
\Delta\dot{\beta} &= -\frac{F_o}{mu^*}C_{y\beta}\Delta\beta + \left(1 - \frac{F_o l}{mVu^*}C_{yr}\right)\Delta r^* - \frac{F_o l}{mVu^*}C_{y\dot{\beta}}\Delta\dot{\beta} - \frac{F_o l}{mVu^*}C_{yp\alpha}p\Delta\alpha \\
&- \frac{F_o l^2}{mV^2u^*}C_{ypq}\,p\Delta q^* - \frac{F_o}{mu^*}C_{y\delta n}\,\Delta\delta_n^* + \frac{F_o}{mu^*}C_{y\beta}\Delta\beta_W - \frac{1}{mu^*}\Delta Y_p^*,
\end{aligned} \tag{25}$$

where one can find, in permanent perturbation components $\Delta Z_p^*, \Delta Y_p^*$, the gravitational terms as well as the residuals from the approximations made after symmetrization of the incidence angle equation α.

We will review Euler dynamic equations, taking into consideration the gyroscopic coupling terms due to Resal trihedral and neglecting the components of the perturbing moment, which are a result of the aerodynamic and gasodynamic asymmetry. Supplementary from the steady roll movement conditions, we will consider that the movement in the vertical symmetry plane is a translation: $q^* = 0$. Meanwhile, in the yaw moment equation, we will neglect the terms of the phugoid mode (ΔV), and in the yaw moment equation we will neglect the coupling terms with the roll movement. The Magnus coupling terms will be outlined, as they appear due to the rotation movement of the missile. Linearizing the last equations (25), we have

$$\begin{aligned}
\Delta\dot{q}^* &= \frac{H_o}{B}C_{m\alpha}\,\Delta\alpha + \frac{H_o l}{BV}C_{mq}\,\Delta q^* + \frac{H_o l}{BV}C_{m\dot{\alpha}}\,\Delta\dot{\alpha} + \frac{H_o l}{BV}C_{mp\beta}\,p\Delta\beta \\
&+ \left(\frac{H_o l^2}{BV^2}C_{mpr} - \frac{A}{B}\right)p\Delta r^* + \frac{H_o}{B}C_{m\delta m}\,\Delta\delta_m^* - \frac{H_o}{B}C_{m\alpha}\,\Delta\alpha_W \\
\Delta\dot{r}^* &= \frac{H_o}{C}C_{n\beta}\Delta\beta + \frac{H_o l}{CV}C_{nr}\,\Delta r^* + \frac{H_o l}{CV}C_{n\dot{\beta}}\Delta\dot{\beta} + \frac{H_o l}{CV}C_{np\alpha}p\Delta\alpha \\
&+ \left(\frac{H_o l^2}{CV^2}C_{npq} + \frac{A}{C}\right)p\Delta q^* + \frac{H_o}{C}C_{n\delta n}\,\Delta\delta_n^* - \frac{H_o}{C}C_{n\beta}\Delta\beta_W\,.
\end{aligned} \tag{26}$$

Noting

$$
\begin{aligned}
a_\gamma^\alpha &= \frac{F_o}{mu^*}C_{z\alpha} = -\frac{F_o}{mu^*}C_{y\beta};\ a_\gamma^q = 1 + \frac{F_o l}{mVu^*}C_{zq} = 1 - \frac{F_o l}{mVu^*}C_{yr}\,; \\
a_\gamma^{\dot\alpha} &= \frac{F_o l}{mVu^*}C_{z\dot\alpha} = -\frac{F_o l}{mVu^*}C_{y\dot\beta};\ a_\gamma^\beta = \frac{F_o lp}{mVu^*}C_{zp\beta} = \frac{F_o lp}{mVu^*}C_{yp\alpha} \\
a_\gamma^r &= \frac{F_o l^2 p}{mV^2u^*}C_{zpr} = \frac{F_o l^2 p}{mV^2u^*}C_{ypq}\,;\ b_\gamma^\delta = \frac{T}{mu^*}; \\
a_\varpi^\alpha &= \frac{H_o}{B}C_{m\alpha} = \frac{H}{C}C_{n\beta};\ a_\varpi^q = \frac{H_o l}{BV}C_{mq} = \frac{H_o l}{CV}C_{nr}\,; \\
a_\varpi^{\dot\alpha} &= \frac{H_o l}{BV}C_{m\dot\alpha} = \frac{H_o l}{CV}C_{n\dot\beta};\ a_\varpi^\beta = \frac{H_o lp}{BV}C_{mp\beta} = -\frac{H_o lp}{CV}C_{np\alpha}\,; \\
a_\varpi^r &= \left(\frac{H_o l^2}{BV^2}C_{mpr} - \frac{A}{B}\right)p = -\left(\frac{H_o l^2}{CV^2}C_{npq} + \frac{A}{C}\right)p; \\
b_\varpi^\delta &= \frac{H_o}{B}C_{m\delta_m} = \frac{H_o}{C}C_{n\delta_n}\,,
\end{aligned}
\tag{27}
$$

for small α and defining complex quantities

$$
\begin{aligned}
\gamma^* &= \alpha - j\beta; \quad \varpi^* = q^* - jr^*; \\
\delta^* &= \delta_m^* - j\delta_n^*; \quad \gamma_W^* = \alpha_W - j\beta_W\,; \\
F_p^* &= Z_p^* + jY_p^*; \quad a_\gamma^\gamma = a_\gamma^\alpha + ja_\gamma^\beta\,; \\
a_\gamma^\varpi &= a_\gamma^q + ja_\gamma^r\,; \quad a_\varpi^\gamma = a_\varpi^\alpha + ja_\varpi^\beta\,; \\
a_\varpi^\varpi &= a_\varpi^q + ja_\varpi^r\,,
\end{aligned}
\tag{28}
$$

where $j = \sqrt{-1}$, previous equations become

$$
\begin{aligned}
\Delta\dot\gamma^* &= a_\gamma^\gamma\,\Delta\gamma^* + a_\gamma^\varpi\,\Delta\varpi^* + a_\gamma^\alpha\,\Delta\dot\gamma^* + b_\gamma^\delta\,\Delta\delta^* - a_\gamma^\alpha\,\Delta\gamma_W^* + \Delta F_p^*/\,(mu^*) \\
\Delta\dot\varpi^* &= a_\varpi^\gamma\,\Delta\gamma^* + a_\varpi^\varpi\,\Delta\varpi^* + a_\varpi^\alpha\,\Delta\dot\gamma^* + b_\varpi^\delta\,\Delta\delta^* - a_\varpi^\alpha\,\Delta\gamma_W^*\,.
\end{aligned}
\tag{29}
$$

Note that for the gasodynamic-controlled missile, the command terms are

$$
b_\gamma^\delta = \frac{T}{mu^*}; \quad b_\varpi^\delta = \frac{l\hat{x}_T T}{B}.
$$

Considering the homogenous form of the equation system, after applying the Laplace transform, one obtains

$$
\begin{bmatrix} \left(1 - a_\gamma^{\dot\alpha}\right)s - a_\gamma^\gamma & -a_\gamma^\varpi \\ -\left(a_\varpi^{\dot\alpha}s + a_\varpi^\gamma\right) & s - a_\varpi^\varpi \end{bmatrix} \begin{bmatrix} \Delta\gamma^* \\ \Delta\varpi^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\tag{30}
$$

In this case, the characteristic polynomial associated with the system of equations can be put in the form

$$P(s) = \left(1 - a_\gamma^{\dot{\alpha}}\right) s^2 - \left(a_\gamma^\gamma + a_\varpi^\varpi + a_\varpi^{\dot{\alpha}} a_\gamma^\varpi - a_\gamma^{\dot{\alpha}} a_\varpi^\varpi\right) s + a_\gamma^\gamma a_\varpi^\varpi - a_\gamma^\varpi a_\varpi^\gamma, \tag{31}$$

or, after separation of complex terms,

$$\begin{aligned} P(s) &= \left(1 - a_\gamma^{\dot{\alpha}}\right) s^2 + \left[a_\gamma^\gamma + a_\varpi^q + a_\varpi^{\dot{\alpha}} a_\gamma^q - a_\gamma^{\dot{\alpha}} a_\varpi^q + j\left(a_\gamma^\beta + a_\varpi^r + a_\varpi^{\dot{\alpha}} a_\gamma^r - a_\gamma^{\dot{\alpha}} a_\varpi^r\right)\right] \\ &+ a_\gamma^\alpha a_\varpi^q - a_\gamma^q a_\varpi^\alpha + a_\gamma^r a_\varpi^\beta - a_\gamma^\beta a_\varpi^r + j\left(a_\gamma^\alpha a_\varpi^r - a_\gamma^q a_\varpi^\beta + a_\varpi^q a_\gamma^\beta - a_\varpi^\alpha a_\gamma^r\right). \end{aligned} \tag{32}$$

For analyzing these expressions as well as for the next ones, it is useful to evaluate the magnitude of main parameters. This will permit retaining only the important terms in approximate relations.

For this, we will define the aerodynamic time $t^* = l/V$, a small parameter, and other nondimensional terms as shown by Hacker (1969): $\tilde{m} = m/(\rho S l)$—reduced mass, or relative density and $i_B = \tilde{m}(i_y/l)^2$—reduced moment of inertia in pitch, where i_y is the inertia radius in respect to pitch axis.

With the notations

$$\omega_p = \frac{A}{2B} p, \quad \phi = \frac{\omega_p}{\Omega}, \quad \eta = \frac{i_b}{\zeta \tilde{m}}, \quad \chi = 2\eta\hat{\omega}_p, \tag{33}$$

where ω_p is the angular speed in precession, and if we consider the case of aerodynamic-stabilized missiles, with low rotation speeds and the pressure center behind the mass center, the characteristic polynomial has the real part of the free term positive. Then,

$$P(s) = s^2 + 2\Omega\left(\xi + j\phi\right) s + \Omega^2 \left(1 + j\chi\right), \tag{34}$$

where Ω and ξ have same meaning as in the case of the stabilized-roll missiles and resulting from

$$\begin{aligned} T^2 &\equiv \frac{1}{\Omega^2} = \frac{1 - a_\gamma^{\dot{\alpha}}}{a_\gamma^\alpha a_\varpi^q - a_\gamma^q a_\varpi^\alpha} \cong \frac{2t^{*2} i_B}{(-C_{m\alpha})}; \quad \eta = \frac{i_B}{\zeta \tilde{m}}; \quad \zeta = \frac{C_{m\alpha}}{C_{z\alpha}} \\ \xi &= \frac{a_\gamma^\alpha + a_\varpi^q + a_\varpi^{\dot{\alpha}} a_\gamma^q - a_\gamma^{\dot{\alpha}} a_\varpi^q}{2T\left(a_\gamma^\alpha a_\varpi^q - a_\gamma^q a_\varpi^\alpha\right)} \cong \frac{1}{\sqrt{8 i_B (-C_{m\alpha})}} \left[\eta\left(-C_{m\alpha}\right) + \left(-C_{mq}\right) + \left(-C_{m\dot{\alpha}}\right)\right]. \end{aligned} \tag{35}$$

The roots of the characteristic polynomial are

$$s_{1,2} = -\xi\Omega - j\omega_p \pm \Omega\sqrt{\xi^2 - 1 - \phi^2 - j\left(\chi - 2\xi\phi\right)}, \tag{36}$$

or, with $\chi \cong 2\xi\phi$:

$$s_{1,2} \cong -\xi\Omega - j\omega_p \pm j\Omega\sqrt{\sigma}, \tag{37}$$

where

$$\sigma = 1 - \xi^2 + \phi^2. \tag{38}$$

These roots, excepting the precession term, have a form close to the one for the rapid movement of a stabilized-roll missile, that is, they are complex numbers, with negative real parts and high values of the modulus.

4. TRANSFER FUNCTIONS OF THE CONTROLLED MISSILE

For defining flight quality indices of the rolling missile, we will review the equations in (29) for the rapid longitudinal movement in the nonhomogenous form:

$$\begin{bmatrix} \Delta\gamma^* \\ \Delta\varpi^* \end{bmatrix} = \mathbf{A}^{-1}(s) \begin{bmatrix} b_\gamma^\delta & -a_\gamma^\alpha & 1/(mu^*) \\ b_\varpi^\delta & -a_\varpi^\alpha & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta^* & \Delta\gamma_W^* & \Delta F_p^* \end{bmatrix}^T, \tag{39}$$

where the inverse is

$$\mathbf{A}^{-1}(s) = \frac{1}{P(s)} \begin{bmatrix} s - a_\varpi^\varpi & a_\gamma^\varpi \\ a_\varpi^{\dot\alpha} s + a_\varpi^\gamma & \left(1 - a_\gamma^{\dot\gamma}\right) s - a_\gamma^\gamma \end{bmatrix}, \tag{40}$$

$P(s)$ being the characteristic polynomial associated to the system given by equations (31), (32), or (34).

For this case, the main transfer functions are

$$\begin{aligned} H_\gamma^\delta(s) &= \frac{b_\gamma^\delta s + a_\gamma^\varpi b_\varpi^\delta - a_\varpi^\varpi b_\gamma^\delta}{P(s)}; \quad H_\gamma^{\gamma_W}(s) = -\frac{a_\gamma^\alpha s + a_\gamma^\varpi a_\varpi^\alpha - a_\varpi^\varpi a_\gamma^\alpha}{P(s)}; \\ H_\gamma^F(s) &= \frac{s - a_\varpi^\varpi}{mu^* P(s)}; \quad H_\varpi^\delta(s) = \frac{\left(b_\varpi^\delta + a_\varpi^{\dot\alpha} b_\gamma^\delta - a_\gamma^{\dot\alpha} b_\varpi^\delta\right) s + a_\varpi^\gamma b_\gamma^\delta - a_\gamma^\gamma b_\varpi^\delta}{P(s)}; \\ H_\varpi^{\gamma_W}(s) &= -\frac{\left(a_\varpi^\alpha + a_\varpi^{\dot\alpha} a_\gamma^\alpha - a_\gamma^{\dot\alpha} a_\varpi^\alpha\right) s + a_\varpi^\gamma a_\gamma^\alpha - a_\gamma^\gamma a_\varpi^\alpha}{P(s)}; \quad H_\varpi^F = \frac{a_\varpi^{\dot\alpha} s + a_\varpi^\gamma}{mu^* P(s)}, \end{aligned} \tag{41}$$

and the system can be put in the form

$$\begin{aligned} \Delta\gamma^* &= H_\gamma^\delta(s)\,\Delta\delta^* + H_\gamma^{\gamma_W}(s)\,\Delta\gamma_W^* + H_\gamma^F(s)\,\Delta F_p^*, \\ \Delta\varpi &= H_\varpi^\delta(s)\,\Delta\delta^* + H_\varpi^{\gamma_W}(s)\,\Delta\gamma_W^* + H_\varpi^F(s)\,\Delta F_p^*. \end{aligned} \tag{42}$$

We will find the rotational speed of the tangent to the trajectory starting from equation (42), to obtain the global structural schema for the guided missile system:

$$\Delta\omega^* = \Delta\varpi^* - \Delta\dot\gamma^*, \tag{43}$$

where

$$\omega^* = \omega_m - j\omega_n , \tag{44}$$

and, taking into account equation (42),

$$\Delta\omega^* = H_\omega^\delta(s)\,\Delta\delta^* + H_\omega^{\gamma W}(s)\,\Delta\gamma_W^* + H_\omega^F(s)\,\Delta F_p^*, \tag{45}$$

in which

$$\begin{aligned} H_\omega^\delta(s) &= H_\varpi^\delta(s) - sH_\gamma^\delta(s)\,; \quad H_\omega^{\gamma W}(s) = H_\varpi^{\gamma W}(s) - sH_\gamma^{\gamma W}(s)\,; \\ H_\omega^F(s) &= H_\varpi^F(s) - sH_\gamma^F(s)\,. \end{aligned} \tag{46}$$

In the following, we will try to simplify the transfer function. Taking into account equation (27), one can read

$$H_\omega^\delta(s) = \frac{-b_\gamma^\delta s^2 + \left[\left(1 - a_\gamma^\varpi - a_\gamma^{\dot\alpha}\right) b_\varpi^\delta + \left(a_\varpi^\varpi + a_\varpi^{\dot\alpha}\right) b_\gamma^\delta\right] s + a_\varpi^\gamma b_\gamma^\delta - a_\gamma^\gamma b_\varpi^\delta}{P(s)},$$

or, after expanding complex terms,

$$\begin{aligned} H_\omega^\delta(s) &= \frac{-b_\gamma^\delta s^2 + \left\{\left[\left(a_\varpi^q + a_\varpi^{\dot\alpha}\right) b_\gamma^\delta - \left(a_\gamma^q + a_\gamma^{\dot\alpha} - 1\right) b_\varpi^\delta\right] + j\left(a_\varpi^r b_\gamma^\delta - a_\gamma^r b_\varpi^\delta\right)\right\} s}{P(s)} \\ &+ \frac{\left(a_\varpi^\alpha b_\gamma^\delta - a_\gamma^\alpha b_\varpi^\delta\right) + j\left(a_\varpi^\beta b_\gamma^\delta - a_\gamma^\beta b_\varpi^\delta\right)}{P(s)}, \end{aligned} \tag{47}$$

where the characteristic polynomial is given by equation (32).

With notations

$$k_\omega^\delta = \frac{a_\varpi^\alpha b_\gamma^\delta - a_\gamma^\alpha b_\varpi^\delta}{a_\varpi^q a_\gamma^\alpha - a_\gamma^q a_\varpi^\alpha}, \quad 2\vartheta\hat\omega_p T_\omega = \frac{a_\varpi^r b_\gamma^\delta - a_\gamma^r b_\varpi^\delta}{a_\varpi^\alpha b_\gamma^\delta - a_\gamma^\alpha b_\varpi^\delta},$$

and disregarding second-order terms

$$k_\omega^\delta = \frac{C_{m\delta m}\left(1 - \zeta\dfrac{C_{z\delta m}}{C_{m\delta m}}\right)}{2t^*\tilde m\zeta}; \quad \vartheta = \zeta\eta\frac{(-C_{z\delta m})}{C_{m\delta m}}; \quad T_\omega \cong \frac{2t^*\tilde m}{(-C_{z\alpha})\left(1 - \zeta\dfrac{C_{z\delta m}}{C_{m\delta m}}\right)}, \tag{48}$$

the transfer function of the angular speed of the tangent to the trajectory can be put in the form

$$H_{\omega}^{\delta}(s) = k_{\omega}^{\delta}\left(1 + 2j\vartheta\hat{\omega}_p T_{\omega} s\right)\left(T^2 s^2 + 2T(\xi + j\phi)s + 1 + j\chi\right)^{-1}, \tag{49}$$

where k_{ω}^{δ} is called the command factor, and T_{ω} is the advance time to command, or the time constant of the plant.

Then, we will search for the connection between incidence (γ^*) and angular speeds (ϖ^*) and (ω^*). For the beginning, the connection between incidence and angular speed will be expressed, using main transfer functions of these parameters to command deflection. We can construct the transfer function

$$H_{\gamma}^{\varpi}(s) = H_{\gamma}^{\delta}(s) / H_{\varpi}^{\delta}(s), \tag{50}$$

which can be approximated by

$$H_{\gamma}^{\varpi}(s) \cong T_{\omega}\left(1 - 2j\vartheta\hat{\omega}_p\right)\left(T_{\omega} s + 1\right)^{-1}, \tag{51}$$

which means the following simplified form:

$$\Delta\gamma^* / \Delta\varpi^* \cong T_{\omega}\left(1 - 2j\vartheta\hat{\omega}_p\right)\left(T_{\omega} s + 1\right)^{-1}. \tag{52}$$

Reviewing equation (44), one can write

$$\Delta\omega^* = \left(1 - s\frac{\Delta\gamma^*}{\Delta\varpi^*}\right)\Delta\varpi^* \cong \left(1 - \frac{T_{\omega}(1 - 2j\vartheta\hat{\omega}_p)s}{T_{\omega} s + 1}\right)\Delta\varpi^* = \frac{1 + 2j\vartheta\hat{\omega}_p T_{\omega} s}{T_{\omega} s + 1}\Delta\varpi_p, \tag{53}$$

which leads us to the desired transfer function:

$$H_{\varpi}^{\omega}(s) \equiv \left(T_{\omega} s + 1\right)\left(1 + 2j\vartheta\hat{\omega}_p T_{\omega} s\right)^{-1}. \tag{54}$$

For the incidence, we can write

$$H_{\gamma}^{\omega}(s) = H_{\gamma}^{\varpi}(s) \cdot H_{\varpi}^{\omega}(s) \cong T_{\omega}\left(1 - 2j\vartheta\hat{\omega}_p\right)\left(1 + 2j\vartheta\hat{\omega}_p T_{\omega} s\right)^{-1}. \tag{55}$$

Finally, we can evaluate the components of the complex acceleration perpendicular to speed, which, if we take into account the small angles hypothesis used for symmetrization of the longitudinal movement, has the directions of y^* and z^* axes and the approximate values of

$$\Delta a^* \cong V\Delta\omega^*, \tag{56}$$

where

$$a^* = -\left(a_z^* + ja_y^*\right). \tag{57}$$

Now the structural scheme of controlled movement for the low rotation speed missile can be built (Figure 2).

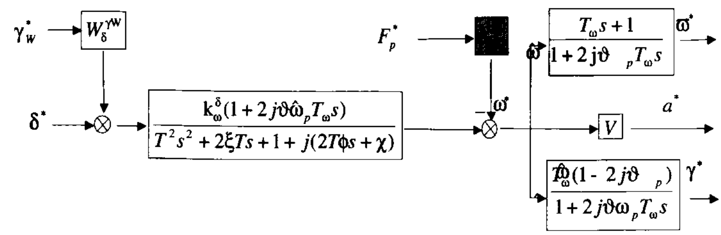

Figure 2. Structural scheme of controlled movement for the low rotation speed missile.

The main stability and control indices, as defined by Niţă et al. (1999b), will be reviewed in the next paragraphs.

For missile sensitivity, we start from the acceleration equation, applying a step input ($\dot{\delta}^* = 0$), which leads to the next form for the equation:

$$\Delta\ddot{a}^* + 2\Omega\,(\xi + j\phi)\,\Delta\dot{a}^* + \Omega^2\,(1 + j\chi)\,\Delta a^* = V\Omega^2 k_\omega^\delta \Delta\delta^*. \tag{58}$$

The solution is

$$\Delta a^* = C_1 e^{-(z_1 - z_2)\,t} + C_2 e^{-(z_1 + z_2)\,t} + \frac{V k_\omega^\delta}{1 + j\chi}\Delta\delta^*. \tag{59}$$

With zero initial conditions,

$$\Delta\dot{a}^*\,(0) = 0; \quad \Delta a^*\,(0) = 0, \tag{60}$$

the solution is

$$\Delta a^* = \frac{V k_\omega^\delta \Delta\delta^*}{1 + j\chi}\left[1 - \frac{e^{-\xi\Omega t} e^{-j\omega_p t}}{\sqrt{\sigma}}\,\mathrm{ch}\left(j\Omega\sqrt{\sigma}t + \psi\right)\right], \tag{61}$$

where ψ is the initial phase.

For the particular case of the movement without rotation, the previous relation became

$$\Delta a^* = V k_\omega^\delta \Delta\delta^*\left[1 - \frac{e^{-\xi\Omega t}}{\sqrt{\sigma}}\cos\left(\Omega\sqrt{\sigma}t - \varphi\right)\right], \tag{62}$$

which is the same as the one for the stabilized-roll missile.

Determining the stabilizing final value of acceleration can make another verification of the above relation. Applying the final value theorem to the transfer function for the step input, the following obtained:

$$\lim_{t\to\infty}\Delta a^*\,(t) = \lim_{s\to\infty}\frac{\Delta\delta^*}{s} s W_a^\delta\,(s) = \frac{V k_\omega^\delta \Delta\delta^*}{1 + j\chi}, \tag{63}$$

a result identical with the one obtained by applying the limiting in equation (61).

Furthermore, expanding these equations, valid in the stabilized regime, we read

$$\left(\Delta a_z^* + j\Delta a_y^*\right) = -\frac{Vk_\omega^\delta (1 - j\chi)}{1+\chi^2}\left(\Delta\delta_m^* - j\Delta\delta_n^*\right), \tag{64}$$

which, after separation of real and imaginary parts, can be put in matrix notation,

$$\begin{bmatrix} \Delta a_z^* \\ \Delta a_y^* \end{bmatrix} = -\frac{Vk_\omega^\delta}{1+\chi^2}\begin{bmatrix} 1 & \chi \\ -\chi & 1 \end{bmatrix}\begin{bmatrix} \Delta\delta_m^* \\ -\Delta\delta_n^* \end{bmatrix}, \tag{65}$$

outlining the coupling between channels due to rotation.

On the other hand, one can express the modulus of the acceleration vector as a function of the modulus of the command vector:

$$|\Delta a^*| = \sqrt{\Delta a_y^{*2} + \Delta a_z^{*2}} = \frac{Vk_\omega^\delta}{1+\chi^2}\sqrt{\left(\Delta\delta_m^{*2} + \Delta\delta_n^{*2}\right)\left(1+\chi^2\right)} = \frac{Vk_\omega^\delta\,|\Delta\delta|}{\sqrt{1+\chi^2}}, \tag{66}$$

which shows the diminishing of the command capacity of the missile due to rotation and permits determination of the sensitivity:

$$\Lambda = \frac{|\Delta a^*|}{\tilde{q}\,|\Delta\delta^*|} = \frac{Vk_\omega^\delta}{\tilde{q}\sqrt{1+\chi^2}}. \tag{67}$$

As regards the other main indices, that is, the reduced amortization factor $\aleph = \xi/\sqrt{\rho}$ and pulsation factor $\Re = \Omega^2/\tilde{q}$, they have the same expression as for the stabilized roll missile.

5. MODEL OF THE GUIDED FLIGHT

For building the model of the guided flight, we will start from the matrix expression of the guidance law, which, taking into account only guidance terms, has the form

$$\begin{bmatrix} u_m \\ -u_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{A}_p\mathbf{A}_M^T\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_y \\ -u_z \end{bmatrix}, \tag{68}$$

where $\mathbf{A}_p$ is the transform matrix between the earth reference trihedral and the mobile trihedral linked to the missile, and $\mathbf{A}_M^T$ is the transpose of the transformation matrix, which links the guidance trihedral to the earth trihedral.

Considering that in the base movement the missile heading is identical with the heading angle of the guidance trihedral, and neglecting the incidence in the vertical plane, to achieve the symmetry of the two guidance planes, the guidance command can be linearized as

$$\begin{bmatrix} \Delta u_m & -\Delta u_n \end{bmatrix}^T = \mathbf{K}_\Phi \begin{bmatrix} \Delta u_y & -\Delta u_z \end{bmatrix}^T, \tag{69}$$

where

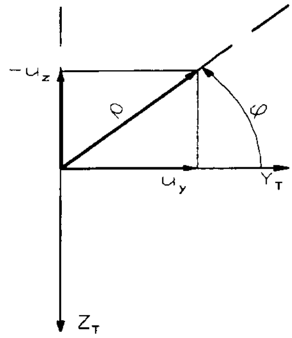

Figure 3. Command in polar coordinates for the case of the angular deviation control.

$$\mathbf{K}_\Phi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{A}_\Phi \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos(\Phi) & -\sin(\Phi) \\ \sin(\Phi) & \cos(\Phi) \end{bmatrix}, \tag{70}$$

and reminding that $\mathbf{A}_\Phi$ is the direct rotation matrix with the lateral inclination angle given by equation (22).

Now we will try to convert matrix expression to complex form using a polar coordinate system capable of revealing the modulus and phase of the command.

If we put the commands in complex form,

$$u = u_m - ju_n; \quad u^* = u_y - ju_z, \tag{71}$$

equation (69) becomes

$$u = u^* e^{j\Phi}. \tag{72}$$

As shown in Figure 3, the guidance command can be expressed in polar coordinates as

$$u^* = \rho e^{j\varphi}, \tag{73}$$

where $\rho = \sqrt{u_y^2 + u_z^2}$ is the modulus of the command and $\varphi = -\text{arctg}(u_z/u_y)$ is the phase of the command. In this case, the guidance command becomes

$$u = \rho e^{j(\Phi+\varphi)}. \tag{74}$$

Most of the single channel missiles do not use deflection proportional to command, adopting the time command as shown by Kazakov and Misakov (1985) and by Niţă and Andreescu (1964). In this way, if the command signal is constant for a time range T, necessary for achieving a certain maneuver, in the case of proportional deflection (Figure 4), a constant level of it in the time range will be kept; in the case of deflection controlled in time (Figure 5), its maximum value in one direction will be kept for a time (T_1), and then in the other direction,

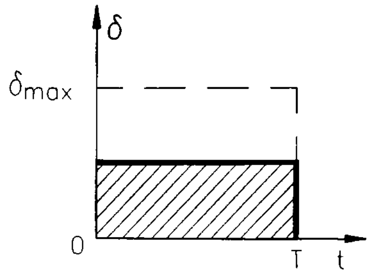

Figure 4. Deflection proportional with time.

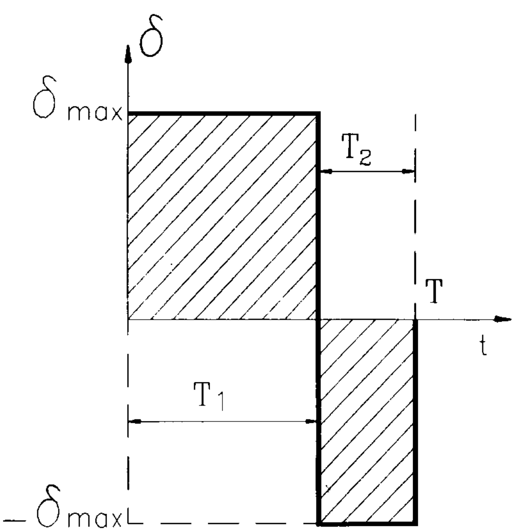

Figure 5. Command for time control.

also at the maximum value, but for a shorter period of time (T_2). Finally, if integrals over T period of the two types of deflections, shown in Figures 4 and 5, are equal, it is accepted that they are equivalent as they lead to equal maneuvers.

For achieving deflection commanded in duration, it is necessary first to express commands in polar coordinates, as shown in Figure 5, considering the Resal trihedral collinear with guidance trihedral, and then to apply them using shapes such as

$$f\left(\tilde{\Phi}, k\right) = 2\sin\tilde{\Phi}\left(k + \cos\tilde{\Phi}\right), \tag{75}$$

functions recommended also by Kocetkov, Polovko, and Ponomarev (1964), where k, called the filling factor, is given by

$$k = \rho / u_{\max}, \tag{76}$$

and $\tilde{\Phi} = \Phi + \varphi + \pi/2 = \omega_x t + \varphi + \pi/2$ is the relative roll angle, with $\varphi = -\mathrm{arctg}(u_z/u_y)$ being the command phase (Figure 3). The above function is defined in the range $\tilde{\Phi} \in [-2\pi; 2\pi]$.

Applying this function is made through a nonlinear element of relay type, the expression for the pitch deflection being

$$\delta_m\left(\tilde{\Phi}, k\right) = \delta_{\max}\cdot \mathrm{sign}\left(f\left(\tilde{\Phi}, k\right)\right). \tag{77}$$

To check the way the function $f(\tilde{\Phi}, k)$ generates the desired deflection, it is necessary to determine the zeros of this function, which correspond to the point of sign changes of the deflection.

For this reason, equation (75) is put in the equation form

$$2\sin\tilde{\Phi}\left(k + \cos\tilde{\Phi}\right) = 0, \tag{78}$$

whose solutions are determined from the conditions $\sin\tilde{\Phi} = 0$ and $\cos\tilde{\Phi} = -k$.

For the interval $[-2\pi, 2\pi]$ in which $\tilde{\Phi}$ takes its values, the points of sign changes due to conditions $\sin\tilde{\Phi} = 0$ are $\tilde{\Phi} = -2\pi, -\pi, 0, \pi, 2\pi$.

From condition $\cos\tilde{\Phi} = -k$, a second group will be obtained. For the case of $k = 1$, it results in $\tilde{\Phi} = -\pi, \pi$, and for $k = 0$, it results in $\tilde{\Phi} = -3\pi/2, -\pi/2, \pi/2, 3\pi/2$.

In the range $[0, 2\pi]$, noting with T_1 the time in which the command plane rotates over circular sector $\tilde{\Phi}_1$,

$$T_1 = \tilde{\Phi}_1 \omega_x^{-1}, \tag{79}$$

and with T_2 the time for rotating over $\tilde{\Phi}_2$, as to complete a semicircle, when a new change in deflection sign occurs,

$$T_2 = \tilde{\Phi}_2 \omega_x^{-1} = \left(\pi - \tilde{\Phi}_1\right)\omega_x^{-1}, \tag{80}$$

the filling factor can be expressed as

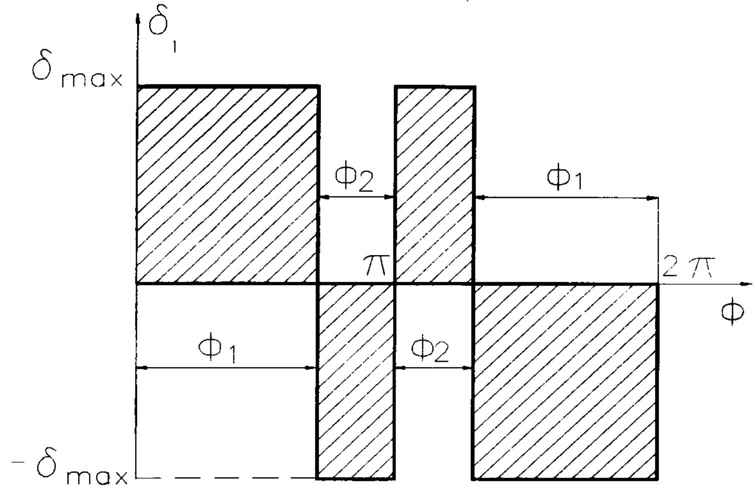

Figure 6. Deflection form for a rotation period of the missile.

$$k = (T_1 - T_2)(T_1 + T_2)^{-1}. \tag{81}$$

This equation is also recommended by Kocetkov, Polovko, and Ponomarev (1964).

In the following, we will determine the equivalent deflection, which, for the particular case when the phase of the command is $\pi/2$ and the advance angle and the response time compensate each other ($\theta = \tau\,\omega_x$), coincides with ideal deflection in pitch. In this case, the deflection obtained by applying equation (77), for a complete missile rotation, has the form in Figure 6. In Figure 7, vectors represent the mode of obtaining the equivalent command force, corresponding to the deflection in Figure 6, for a complete rotation of the missile.

For the general case, the complex instantaneous equivalent deflection is

$$\tilde{\delta} = \delta_m\, e^{-j(\Phi+\theta-\tau\omega_x)} = -j\delta_m\, e^{-j\tilde{\Phi}}\, e^{(\varphi+\theta-\tau\omega_x)}, \tag{82}$$

where, as we have noted, θ is the angle of command advance and τ is the delay time of the command. The medium equivalent deflection for a whole period is, in this case,

$$\delta^* = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\delta}\,\mathrm{d}\tilde{\Phi} = \frac{1}{2\pi} e^{-j(\varphi+\theta-\tau\omega_x)} \int_0^{2\pi} \delta_m\, e^{-j\tilde{\Phi}}\,\mathrm{d}\tilde{\Phi}. \tag{83}$$

Taking into account the parity of the command for a whole rotation, the medium equivalent deflection becomes

$$\delta^* = e^{j(\varphi+\theta-\tau\omega_x)}\,\frac{1}{2\pi}\int_0^{2\pi} \delta_m \sin\tilde{\Phi}\,\mathrm{d}\tilde{\Phi}. \tag{84}$$

The integral in the above equation can be evaluated using the diagram in Figure 6.

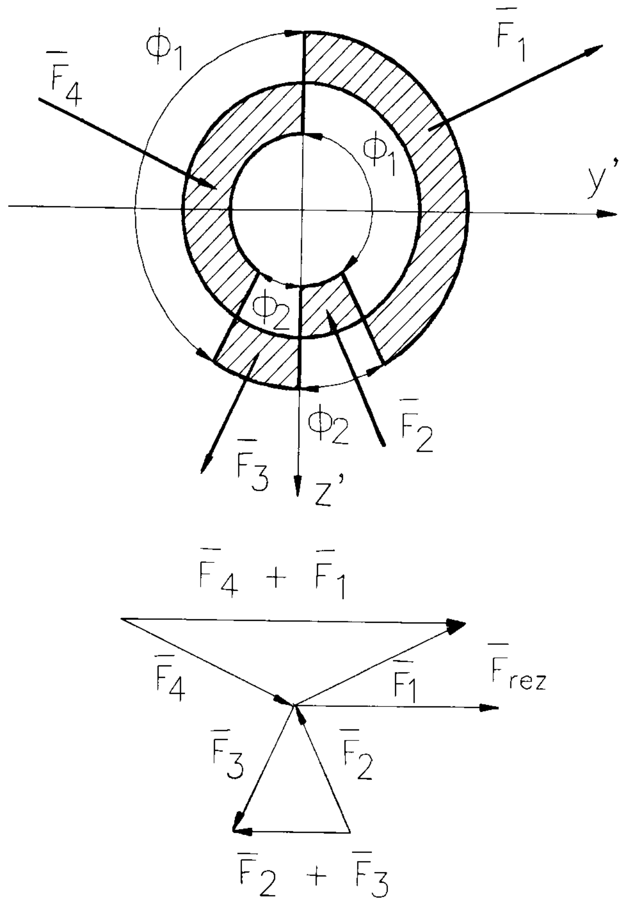

Figure 7. Resultant command force for a complete missile rotation.

$$\frac{1}{2\pi}\int_{0}^{2\pi} \delta_m \sin\tilde{\Phi} d\tilde{\Phi} = -\frac{2\delta_{\max}}{\pi}\cos\tilde{\Phi}_1 = \frac{2\delta_{\max}}{\pi}k, \tag{85}$$

so the medium equivalent deflection is

$$\delta^* = \frac{2\delta_{\max}}{\pi} k e^{j(\varphi+\theta-\tau\omega_x)} = \frac{2\delta_{\max}}{\pi u_{\max}} e^{j(\theta-\tau\omega_x)} u^*, \tag{86}$$

from which one can determine the variation form

$$\Delta\delta^* = k_\delta^u e^{j(\theta-\tau\omega_x)} \Delta u^*, \tag{87}$$

where

$$k_\delta^u = \frac{2\delta_{\max}}{\pi u_{\max}}. \tag{88}$$

In Figure 7 is presented the way of building the command force for a complete rotation. The command force generates a command moment, which, depending on the elevators or mobile nozzle position in respect to the center of mass, can have an opposite direction. To decide, check the standard described in Niţă et al. (1999a), which defines the positive deflection in respect to any axes, as per positive command moment with respect to that axis.

To build the structural scheme, in addition to the description of the command system, we need to define the complex form of some pairs of parameters in the guidance plane.

In the case of the self-guided missile, for the absolute angles of the firing line, we define the complex value as

$$\sigma^* = \sigma_y - j\sigma_z, \tag{89}$$

and for the absolute angles of the speed:

$$\vartheta^* = \gamma_M - j\chi_M. \tag{90}$$

At last, we define the complex form of the input function:

$$f^* = f_y + jf_z. \tag{91}$$

Adding the transfer function of the gyro-coordinator,

$$H_u^v(s) = sk_u^\omega (\tau_c s + 1)^{-1}, \tag{92}$$

and the transfer functions of the kinematics block, respectively, for equations in accelerations or speeds,

$$H_C = \frac{1}{(t_f s - 2)(1 - \tilde{p})}; \quad H_C = \frac{1}{(t_f s - 1)(1 - \tilde{p})}, \tag{93}$$

where

$$\tilde{p} = (V_T \cos\mu_T)(V_M \cos\mu_M)^{-1}; \quad t_f = R(-\dot{R})^{-1},$$

we can build the structural schema for the self-guided missile with mono-channel rotation, as seen in Figure 8.

For the remote-guided missile, we define the following complex for the absolute angles of the guidance line:

$$\varphi^* = \varphi_y - j\varphi_z. \tag{94}$$

Similarly, for the absolute angles of the firing line, we define

$$\sigma^* = \sigma_y - j\sigma_z, \tag{95}$$

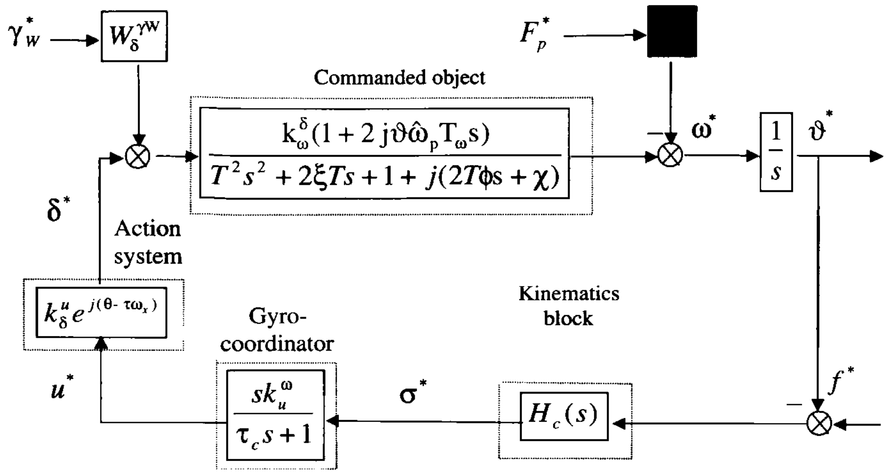

Figure 8. Structural schema for self-guided slow rotation mono-channel missile.

and for the angular deviations in the guidance planes, we obtain

$$\varepsilon^* = \varepsilon_y - j\varepsilon_z, \tag{96}$$

where

$$\varepsilon_y = \sigma_y - \varphi_y; \quad \varepsilon_z = \sigma_z - \varphi_z.$$

Between the three predefined quantities, the following relationship is defined:

$$\varepsilon^* = \sigma^* - \varphi^*. \tag{97}$$

Finally, the complex form of the enter function of zero order in the system is due exclusively to the operator or the targeting system

$$f_0^* = f_{0y} + jf_{0z}, \tag{98}$$

which eventually might be accumulated at the entrance of the absolute angle of the targeting line (σ^*).

Regarding the processing of the error signal and the concept of the guided command, this one is realized with an element of PD type (proportional derivative), which has the following known structure:

$$H_u^\varepsilon(s) = P(k_u^\varepsilon + sk_u^\nu)(\tau_1 s + 1)^{-1}, \tag{99}$$

where k_u^ε and k_u^ν are guidance constants.

In some cases, the direction may be realized manually with the help of a human operator, as indicated by Kocetkov, Polovko, and Ponomarev (1964), and it might be modeled with a transfer function of the following form:

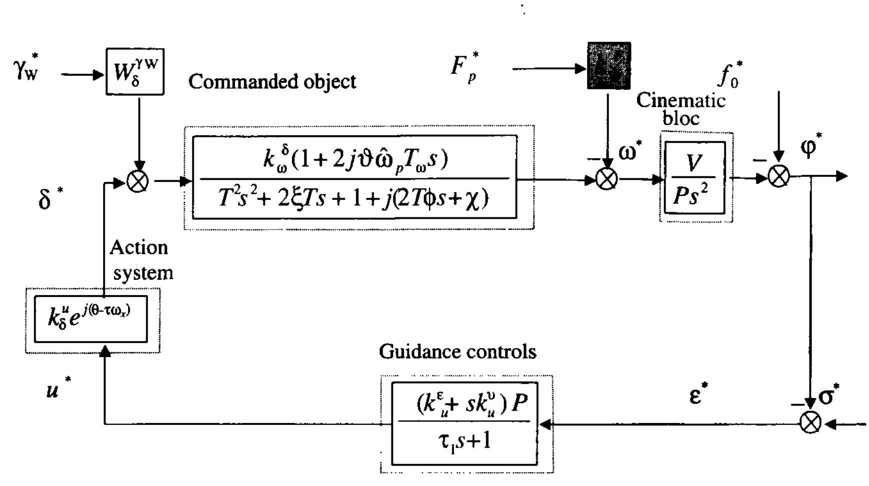

Figure 9. Structural scheme for the remote-guided semiautomated missile with slow mono-channel rotation.

$$H_o(s) = e^{-\tau_0 s}(T_o s + 1)^{-1}, \tag{100}$$

which, as indicated in the same reference, may be approximated with

$$W_o(s) = \left(1 - s\frac{\tau_0}{3}\right)^3 (T_o s + 1)^{-1}. \tag{101}$$

Based on the previous relations, the structural scheme for the missile with semiautomated, remote-guided mono-channel rotation for the angular deflection control in the case of the fix carrier can be built as in Figure 9.

6. EXAMPLE OF CALCULATION

Opposite to the flying machines with stabilized roll where the analysis is done after decoupling the channels, for the missile with rotation, the analysis of the scheme is done simultaneously on both channels, as the used terms are complex terms. The complex terms given in the scheme shown in Figure 9 represent the coupling terms due to the rotation, and they are specific to this type of missile.

Furthermore, there will be presented some characteristics of this type of missile by analyzing a model of calculation close to the one of an antichariot missile with gasodynamic control. The main model parameters, as given in the scheme from Figure 9, are the following:

$$\begin{aligned}
V &= 120\,m/s; \quad \omega_x = 54\,s^{-1}; \quad k_\omega^\delta = 0{,}673\,s^{-1};\\
T &= 0{,}133\,s; \quad \xi = 0{,}229; \quad \phi = 0{,}115; \quad \vartheta = -0{,}153; \quad T_\omega = 1\,s;\\
\hat{\omega}_p &= 0{,}009; \quad \chi = 0{,}06; \quad \theta = 0{,}538; \quad \tau = 0{,}011\,s; \quad \tau_1 = 0{,}2\,s;\\
k_u^\varepsilon &= 0{,}01; \quad k_u^\nu = 0{,}0085\,s; \quad k_\delta^u = 0{,}5.
\end{aligned}$$

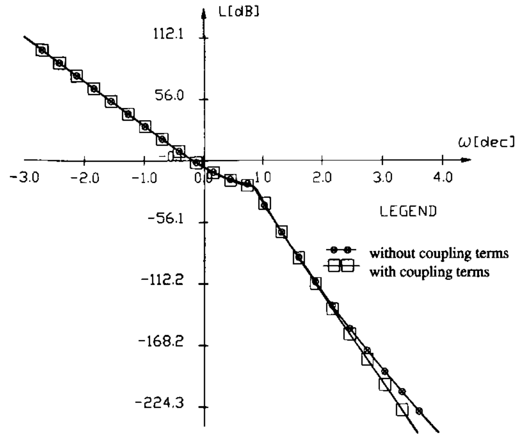

Figure 10. Logarithmic magnitude-frequency characteristics.

For the model of calculation considered, by applying the general methods of analysis in the frequency domain, there will be realized a comparative presentation of the main characteristics with and without the coupling terms.

Bode diagrams are presented in Figures 10 and 11. In Figure 10, there is presented the logarithmic characteristics of magnitude in frequency. It can be seen that the cut frequency may be found on the line of slope -40 dB/dec, the coupling is found in the high magnitudes domain, being in this way damped. In Figure 12, the characteristic of phase magnitude is presented, as an increase of the coupling at high magnitudes is noticed. By taking into account that the magnitude of the missile's pulsation increases with the square of the speed, in the absence of a corresponding damping, at high speeds a phenomenon of accentuated precession may be explained by the approach of the resonance zone of the longitudinal mode to the one of roll; this resonance may appear through a transfer of energy between channels. Fortunately, with the increasing of the frequency, the magnitude decreases rapidly, with the slope of -60 dB/dec, which leads to a strong damping. However, an accentuated speed increase over certain limit values is not wished and may lead to the appearance of an instability phenomenon.

From the diagram of magnitude frequency (Figure 11) is noticed the frequency of resonance, where the characteristic has a local maximum of approximately $8[1/s]$. The cutting frequency, where the characteristic of magnitude frequency intersects the axis of frequencies, is of $\omega = 0,8[1/s]$. From the characteristics of phase frequency, it is noticed that the point where the phase characteristic intersects the line at $-180°$ is $\omega_\pi = 10[1/s]$, which is very small over the maximum local frequency. In this case, the magnitude margin, which is the value with a different sign of the magnitude measured in dB for the frequency ω_π at which the phase intersects the line at $-180°$ has the value of 30 dB, and the phase margin, corresponding to the cutting frequency, has the value of $-150°$. By summarizing, we can see that the Bode stability criteria are verified. In conformity with this criteria, the condition for a system to be stable is the phase-frequency characteristic must intersect the ω axis in

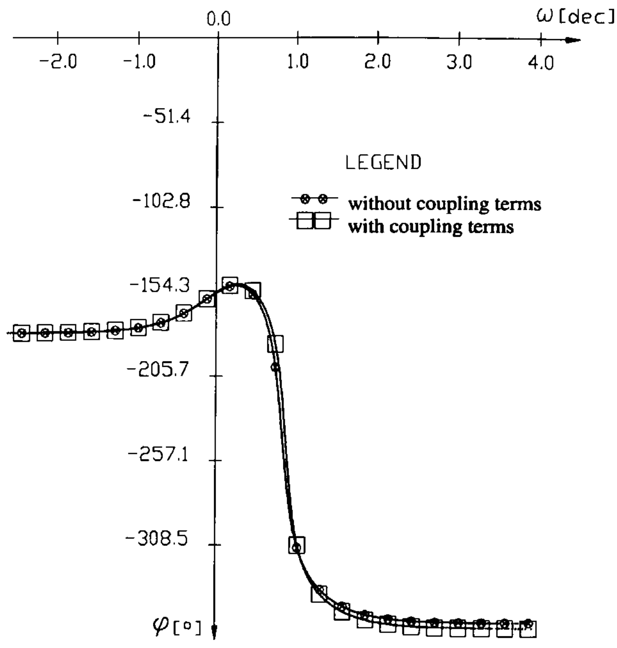

Figure 11. Phase-frequency characteristics.

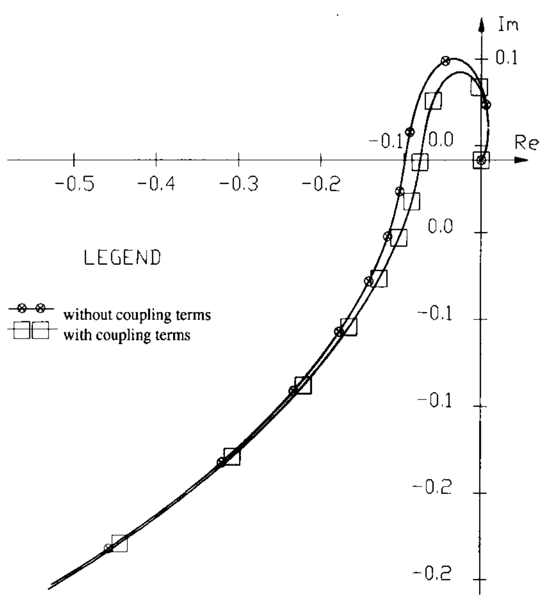

Figure 12. Nyquist diagram.

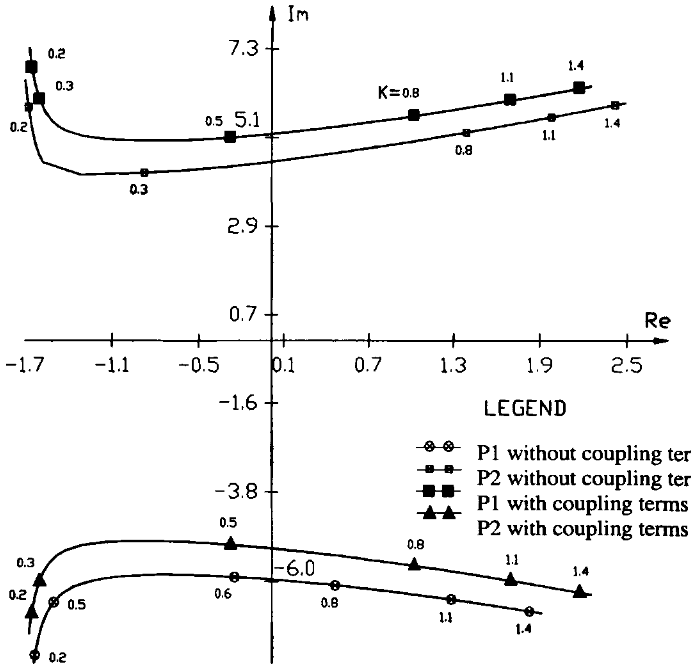

Figure 13. Pole-placement of the rapid motion.

a point after the intersection with the same axis of the magnitude-frequency characteristic ($\omega_\pi > \omega_0$).

In Figure 12, the Nyquist characteristics are presented. According to the stability Nyquist criteria, in cases where the transfer function of the direct route has no poles with positive real part, the closed system is stable if the place of transfer of the open system goes to the right of the critical point $(1, 0j)$. In Figure 12, it may be seen that the stability Nyquist criteria is verified. The coupling terms appear in the domain of high frequencies. Due to the coupling, the diagram is not symmetric with respect to the real axis, as the construction of the negative frequency branch is necessary.

In Figure 13, there exist in an Evans diagram the first two poles of the system as functions of the magnitude in the open system. These poles correspond to the rapid mode of the controlled object, and they have the real part increasing with the amplification coefficient. So that, as seen, high amplification may lead to poles with positive real parts, which means that the system becomes unstable. An asymmetry of the diagram may be observed with respect to the real axis, the asymmetry being due to the coupling terms. It may be observed that the value of magnitude does not modify the influence of the coupling for these two poles.

In Figure 14, there are seen the poles that come from the kinematics bloc, and which initially have a zero value. With the increase of magnitude, the real part of these two poles will diminish, the system being more and more stable. Also an asymmetry of diagram may be observed with respect to the real axis, especially for high values of the magnitude.

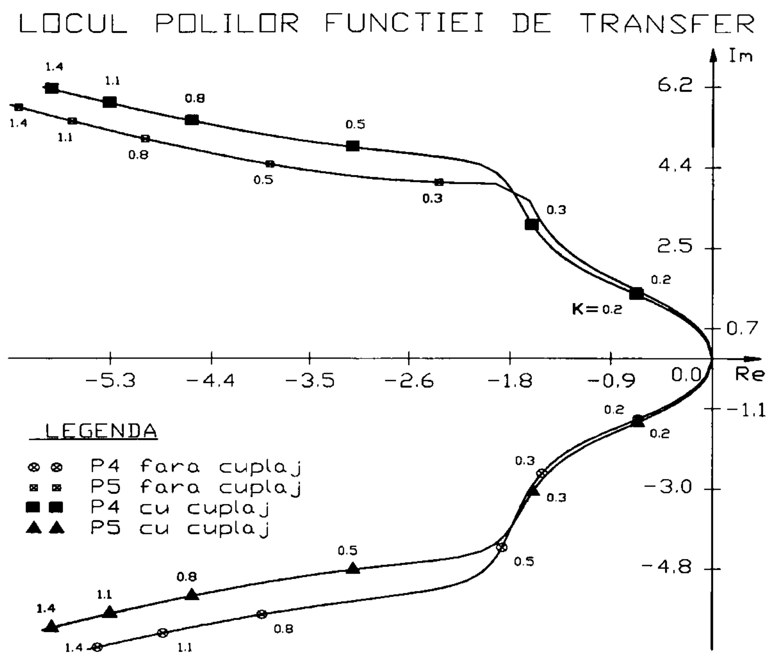

Figure 14. Transfer function pole placement.

Besides the two pairs of poles presented before, there still exists a real pole, which comes from the control system placed in the firing host. It may be seen that, with the increasing of the magnitude, this real pole increases initially, and it stabilizes around -1.4.

In conclusion, a value of the magnitude may be chosen on the direct way, which may ensure a good behavior of the system from the point of view of all magnitudes of system and implicitly of the poles.

NOTE

1. The perturbations caused by the wind can introduce only supplementary angles of incidence; the velocity module augmentation, being specific to the slow motion, can be neglected.

REFERENCES

Carpantier, R., 1989a, *Guidance des avions et des missiles aerodynamiques*, Tome I [Course notes], École Nationale Supérieure de l'Aéronautique et de l'Espace, Toulouse, France.

Carpantier, R., 1989b, *Méthodes de conduite mécanique du vol et pilotage appliques aux missiles*, Tome II [Course notes], École Nationale Supérieure de l'Aéronautique et de l'Espace, Toulouse, France.

Carpantier, R., 1989c, *Autoguidage*, Tome III [Course notes], École Nationale Supérieure de l'Aéronautique et de l'Espace, Toulouse, France.

Hacker, T., 1969, *Stabilitate şi comanda în teoria zborului*, Academiei, Bucuresti, Romania.

Kazakov, I. E. and Misakov, A. F., 1985, *Aviaţionnîe upravliaemîe raketî*, VVIA "N.E. Jukovski," U.R.S.S.

Kocetkov, V. T, Polovko, A. M., and Ponomarev, B. M., 1964, *Teoria sistem teleupravlenia i samonavedenia raket*, Nauka, Moskva, USSR.

Kuzovkov, N. T., 1976, *Sistemî stabilizaţii letatelnîh apparatov, balisticeschih i zenitnîh raket*, Vîsşaia Şkola, Moskva, U.R.S.S.

Niţă, M. M. and Andreescu, D. Şt., 1964, *Zborul rachetei*, Militara, Bucureşti, Romania.

Niţă, M. M., Stiharu, D., Danaila, S., Parvu, P., Theodorescu, B., Chelaru, V., Zafiu, V., and Lascu, P., 1999a, *S.T.P.M.* 40406-99 -*Sistemul racheta comandata*, Bucureşti, Romania.

Niţă, M. M., Stiharu, D., Danaila, S., Parvu, P., Theodorescu, B., Chelaru, V., Zafiu, V., and Lascu, P., 1999b, *S.T.P.M.* 040421-99 -*Indici de calitate a zborului rachetei comandate*, Bucureşti, Romania.

Niţă, M. M., Stiharu, D., Danaila, S., Parvu, P., Theodorescu, B., Chelaru, V., Zafiu, V., and Lascu, P., 1999c, *S.T.P.M.* 40455-99 -*Sistemul racheta dirijata*, Bucureşti, Romania.