STRUCTURED CONTROLLER DESIGN FOR OPTIMAL MODEL MATCHING

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Abstract: In this paper, a BB algorithm is used to designed structured controller that minimizes a matching model criteria. This non convex optimization problem is formulated as a BMI problem with additional variables allowing lower and upper bound estimation by LMI subproblems. To demonstrate the effectiveness of the approach, this global optimization method is used to tune optimally the PID controller gains.

Keywords: structured controller, non convex optimization, model matching, PID controller.

1.INTRODUCTION

Over the past decades, many advances have been made in the field of linear optimal control. In particular, several methods based on Linear Matrix Inequalities (LMI) formulation have been proposed to solve multiple optimal control problems for state feedback and full-order output feedback controllers [1]. Recently, optimal performances of low-order output feedback and static output feedback controllers have been extensively studied [2, 3] . For static output feedback, fixed order controllers and structured controller, the design is also a non convex optimization problem. This problem can be transformed into a Bilinear Matrix Inequality (BMI) optimization problem [4]. The solution can then be solved by using local minimization approach [4] or the branch and bound (BB) algorithm that allow to find a global solution [5] [6].

In this paper, a BB algorithm is used to designed a structured controller [4] that minimizes a matching model criteria. This class of controller is very important for industrial application since its order and its structure can be imposed arbitrary. The optimization problem is formulated as a BMI problem with the additional variables that allow us to formulate the LMI subproblems associated to the lower and the upper bound estimations. Those estimations are necessary to formulate the associated BB algorithm.

This paper is organized as follows. The system model is presented in section 2. Section 3 presents the optimization problem formulation. In section 4, the optimization approach is applied to tune optimally PID controller gains for arbitrary plant, sensors and reference model.

2.SYSTEM MODEL

Continuous linear invariant feedback control systems are generally characterized by a plant, that is the system to be controlled, sensors, and a feedback controller. In this paper, it is assumed that the controller is structured. The gains adjustment that minimizes the difference between a reference model and the closed loop of the feedback control system is then a non convex optimization problem. The blocks-diagram corresponding to this complete system is illustrated by Figure 1. As it can be seen, to be more flexible, the error between the reference model and the closed loop system is weighted by a weight error model. Each block of the diagram in Figure 1 is modelized in its state space form. The reference model is described as

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{r} \\ \mathbf{y}_r &= \mathbf{C}_r\mathbf{x}_r \end{aligned} \tag{1}$$

where $\mathbf{x}_r \in \mathbf{R}^{n_r}$, $\mathbf{y}_r \in \mathbf{R}^{n_y}$ and $\mathbf{r} \in \mathbf{R}^{n_y}$. The plant model is described as

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{A}_p\mathbf{x}_p + \mathbf{B}_p\mathbf{u} \\ \mathbf{y}_p &= \mathbf{C}_p\mathbf{x}_p \end{aligned} \tag{2}$$

where $\mathbf{x}_p \in \mathbf{R}^{n_p}$, $\mathbf{y}_p \in \mathbf{R}^{n_y}$ and $\mathbf{u} \in \mathbf{R}^{n_u}$. Then, the sensors model is described as

$$\begin{aligned} \dot{\mathbf{x}}_s &= \mathbf{A}_s\mathbf{x}_s + \mathbf{B}_s\mathbf{y}_p \\ \mathbf{y}_s &= \mathbf{C}_s\mathbf{x}_s + \mathbf{D}_s\mathbf{y}_p \end{aligned} \tag{3}$$

where $\mathbf{x}_s \in \mathbf{R}^{n_s}$ and $\mathbf{y}_s \in \mathbf{R}^{n_y}$. The weight error model is described as follows

$$\begin{aligned} \dot{\mathbf{x}}_w &= \mathbf{A}_w\mathbf{x}_w + \mathbf{B}_w\varepsilon \\ \mathbf{e} &= \mathbf{C}_w\mathbf{x}_w \end{aligned} \tag{4}$$

where $\varepsilon = \mathbf{y}_r - \mathbf{y}_p = \mathbf{C}_r\mathbf{x}_r - \mathbf{C}_p\mathbf{x}_p$, $\mathbf{x}_w \in \mathbf{R}^{n_w}$ and $\mathbf{e} \in \mathbf{R}^{n_y}$. Finally, the structured controller model is described as

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{A}_c(\mathbf{k})\mathbf{x}_c + \mathbf{B}_c(\mathbf{k})(\mathbf{r} - \mathbf{y}_s) \\ \mathbf{u} &= \mathbf{C}_c(\mathbf{k})\mathbf{x}_c + \mathbf{D}_c(\mathbf{k})(\mathbf{r} - \mathbf{y}_s) \end{aligned} \tag{5}$$

where $\mathbf{x}_c \in \mathbf{R}^{n_c}$ and [4]

$$\begin{bmatrix} \mathbf{A}_c(\mathbf{k}) & \mathbf{B}_c(\mathbf{k}) \\ \mathbf{C}_c(\mathbf{k}) & \mathbf{D}_c(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{c0} & \mathbf{B}_{c0} \\ \mathbf{C}_{c0} & \mathbf{D}_{c0} \end{bmatrix} + \mathbf{\Theta}_L\mathbf{K}_s(\mathbf{k})\mathbf{\Theta}_R \tag{6}$$

with

$$\mathbf{\Theta}_L\mathbf{K}_s(\mathbf{k})\mathbf{\Theta}_R = \sum_{i=1}^{n_k} \mathbf{\Theta}_{Li} k_i \mathbf{\Theta}_{Ri} \tag{7}$$

where $\mathbf{k} \in \mathbf{R}^{n_k}$ is the gain vector, $\mathbf{\Theta}_{Li} \in \mathbf{R}^{n_u \times n_{k_i}}$ and $\mathbf{\Theta}_{Ri} \in \mathbf{R}^{n_{k_i} \times n_s}$ are full rank matrices. The system described by equations (1), (2), (4), (5) and (6) can be rewritten in one state space model as

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathcal{A} + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{C}_u)\mathbf{x} + (\mathcal{B}_r + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{D}_u)\mathbf{r} \\ \mathbf{e} &= \mathcal{C}\mathbf{x} \end{aligned} \tag{8}$$

3.OPTIMIZATION APPROACH

According to the closed loop system described by equation (8), the objective is to find the controller gain vector $\mathbf{k}$ to minimize the error $\mathbf{e}$. This problem can be formulated as

$$\mathbf{k} = \arg\min_{\mathbf{k}} \|\mathbf{H}_{\varepsilon r}\|_2 \quad (9)$$

where $\mathbf{H}_{\varepsilon r}(s)$ is the transfer function of the closed loop system described by equation (8) and $\|\mathbf{H}\|_2$ is the 2-norm of system $\mathbf{H}$ that exist only if $\mathbf{H}$ is Hurwitz-stable. if $\mathbf{H}$ is Hurwitz-stable, $\|\mathbf{H}\|_2$ is defined by [1]

$$\|\mathbf{H}\|_2^2 = \frac{1}{2\pi}\ \mathtt{trace} \int_{-\infty}^{\infty} \mathbf{H}(j\omega)\mathbf{H}(j\omega)^* d\omega \quad (10)$$

To avoid the integral evaluation, the minimization problem described by equations (9) can be rewritten as [1]

$$\mathbf{k} = \arg\min(\ \mathtt{trace}(\mathcal{L}_1(\mathcal{B}_r + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{D}_u, \mathbf{P}))) \quad (11)$$

where $\mathbf{P}$, a symmetric semi-positive definite matrix, is the solution of the following equations :

$$\mathcal{L}_2((\mathcal{A} + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{C}_u), \mathbf{P}) + \mathbf{C}^T\mathbf{C} = \mathbf{0}, \quad (12)$$

$$\mathcal{L}_1(\mathbf{B}, \mathbf{P}) = \mathbf{B}^T\mathbf{P}\mathbf{B} \quad (13)$$

and

$$\mathcal{L}_2(\mathbf{A}, \mathbf{P}) = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} \quad (14)$$

Even though the problem described by equations (11) to (14) is non convex, BB algorithm can be used to find its global solution. The solution $\mathbf{k}$ is then searched in a hypercube $\mathcal{Q}$ [7]. The BB formulation consists of finding a lower bound estimation and an upper bound estimation of the objective function when the decision variables are constrained in a hypercube $\mathcal{Q}$. The BB algorithm is based on the hypercube subdivision (branch) and on the evaluation of lower and upper bounds (bound) in the subdivided hypercubes $\mathcal{Q}_l$. The BB algorithm converges to the global optimal solution if the lower and the upper estimations converge to the same result while the hyper-volume of the subdivided hypercubes $\mathcal{Q}_l$ converges to zero. This condition ensures a global convergence of the algorithm. However, if the distance between the lower and the upper bound is very high until the hyper-volume of the hypercubes $\mathcal{Q}_l$ is very close to zero, the convergence is very slow. It is thus very important to find a lower and an upper bound estimations that are as close as possible to each other.

In this paper, the minimization problem described by (11) to (14) will be transformed into a Bilinear Matrix Inequalities (BMI). Then, a BB algorithm will be used with an LMI minimization problem for the lower and upper bound estimation. This approach has been presented in [5] and has been modified for different class of problems in [6].

4.BMI FORMULATION

The minimization problem described by equations (11) to (14) can be transformed in a BMI formulation [1]:

$$\mathbf{k} = \arg\min(\ \mathtt{trace}(Z)) \quad (15)$$

satisfying

$$\begin{array}{c} \mathbf{P} > 0 \\ \mathcal{L}_2((\mathcal{A} + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{C}_u), \mathbf{P}) + \mathbf{C}^T\mathbf{C} < \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & * \\ (\mathcal{B}_r + \mathcal{B}_u\mathbf{K}_s(\mathbf{k})\mathcal{D}_u)^T\mathbf{P} & Z \end{bmatrix} > 0 \end{array} \quad (16)$$

where the symbol * is used to simplify the writing since matrices are symmetric. BMI formulation described by equations (15) and (16) cannot be transformed into the LMI form by using a change of variable since feedback controller of Figure. 1 is not a state feedback controller nor a full-order output feedback controller. As proposed in [5], BMI optimization problem described by equations (15) and (16) can be solved by using the BB algorithm. The key to this approach is to find LMI optimization problems to estimate an upper and a lower bound of BMI problem in an hypercube $\mathcal{Q}_l$. To accomplish this formulation, the BMI problem is first transformed according to an additional decision variables and the hypercube constraint is added. The additional variables are defined as [6]:

$$\mathbf{W}_i = k_i\mathbf{P}\mathcal{B}_u^i,\ i = 1...n_k \quad (17)$$

where k_i are the component of the gain vector $\mathbf{k}$ and, according to equation (6), $\mathcal{B}_u^i$ is defined as:

$$\mathcal{B}_u^i = \mathcal{B}_u\mathbf{\Theta}_{Li}. \quad (18)$$

In [6], the hypercube constraint is added to the decision variables given by different elements of vector $\mathbf{k}$ and matrix $\mathbf{P}$. In our formulation problem, the different elements of matrix $\mathbf{P}$ are constrained to ensure its positivity. Moreover, the additional variables described by equation (17) are different of the ones proposed in [6]. In fact, in [6], the different elements of matrix $\mathbf{P}$ are all considered as additional variables while in the additional variables described by equation (17), only the different elements of product $\mathbf{P}\mathcal{B}_u^i$ are considered. Now, the additional variables must be incorporated in a new optimization problem formulation. The following proposition allows us to perform an appropriate transformation of the constraints to incorporate the additional variables.

Proposition 1 *If the hypercube $\mathcal{Q}_l$ is defined as follows*

$$\mathcal{Q}_l = \{\mathbf{k} | \underline{k}_i^l < k_i < \overline{k}_i^l,\ i = 1...n_k\}, \quad (19)$$

$n_{k_i} \leq (n+1)/2$ *and* $\mathcal{B}_u^i$ *are full rank matrix, Then, the following constraints set*

$$\mathbf{W}_i = k_i\mathbf{P}\mathcal{B}_u^i,\ i = 1...n_k \quad (20)$$

$$\mathbf{P} > 0 \quad (21)$$

$$\mathbf{k} \in \mathcal{Q}_l \quad (22)$$

is equivalent to the constraints set described by equations (20), (21) and the following one

$$\begin{bmatrix} \tilde{k}_i^l \mathcal{B}_u^{iT} \mathbf{P} \mathcal{B}_u^i & * \\ \mathbf{W}_i - \hat{k}_i^l \mathbf{P} \mathcal{B}_u^i & \tilde{k}_i^l \mathbf{P} \end{bmatrix} > 0, \; i = 1...n_k \tag{23}$$

where $\tilde{k}_i^l = \frac{1}{2}(\overline{k}_i^l - \underline{k}_i^l)$ *and* $\hat{k}_i^l = \frac{1}{2}(\overline{k}_i^l + \underline{k}_i^l)$.

Proof: **Proposition 1** can be proved by using some calculus arguments and the Schur complement. This proposition can be used to transform the BMI problem described by equations (15), (16), (17) and (23) into the following equivalent form:

$$\mathbf{K} = \arg\min(\; \mathtt{trace}(Z)) \tag{24}$$

satisfying constraints given by equations (20), (21), (23) and

$$\mathcal{L}_2(\mathcal{A}, \mathbf{P}) + \sum_{i=0}^{n_k} \mathcal{L}_2(\mathbf{\Theta}_{Ri} \mathcal{C}_u, \mathbf{W}_i) + \mathbf{C}^T \mathbf{C} < \mathbf{0} \tag{25}$$

$$\begin{bmatrix} \mathbf{P} & * \\ (\mathbf{P}\mathcal{B}_r + \sum_{i=0}^{n_k} \mathbf{W}_i \mathbf{\Theta}_{Ri} \mathcal{D}_u)^T & Z \end{bmatrix} > 0 \tag{26}$$

As in [7], the upper bound can be found by considering the 2-norm at the center of the hypercube. By fixing the matrix gains at the center of the hypercube, the BMI described by equations (15) and (16) is transformed into LMI form:

$$UB(\mathcal{Q}_l) = \min(\; \mathtt{trace}(Z)) \tag{27}$$

satisfying

$$\begin{gathered} \mathbf{P} > 0 \\ \mathcal{L}_2((\mathcal{A} + \mathcal{B}_u \mathbf{K}_s(\hat{\mathbf{k}}^l) \mathcal{C}_u), \mathbf{P}) + \mathbf{C}^T \mathbf{C} < \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & * \\ (\mathcal{B}_r + \mathcal{B}_u \mathbf{K}_s(\hat{\mathbf{k}}^l) \mathcal{D}_u)^T \mathbf{P} & Z \end{bmatrix} > 0 \end{gathered} \tag{28}$$

where $\hat{\mathbf{k}}^l$ is the gain vector $\mathbf{k}$ fixed at the center of hypercube $\mathcal{Q}_l$. For the lower bound estimation, the non convex constraints described by equations (20), (21), (23), (25) and (26) is transformed to a convex one by simply removed the equality constraint (20):

$$LB(\mathcal{Q}_l) = \min(\; \mathtt{trace}(Z)) \tag{29}$$

satisfying constraints given by equations (21), (23), (25), (26) and

$$\mathcal{L}_2(\mathcal{A}, \mathbf{P}) + \sum_{i=0}^{n_k} \mathcal{L}_2(\mathbf{\Theta_{Ri}} \mathcal{C}_u, \mathbf{W}_i) + \mathbf{C}^T \mathbf{C} > -\xi \mathbf{I} \tag{30}$$

where ξ is a small numerical value. Notice that the constraint described by equation (30) has been added to allow a better estimation of the lower bound. This constraint restricts the solution of the lower bound estimation problem but does not change the original BMI problem as described by equations (24), (20), (21), (23), (25) and (26).

5.BRANCH AND BOUND ALGORITHM

According to upper and lower bound estimation given by equations (27) to (28) and (29), (21), (23), (25), (26) and (30) , the BB algorithm allowing the solution of the BMI problem can be stated as follows [6]

Algorithm 1

Step 0) Start with a tolerance factor ϵ and the initial hypercube $\mathcal{Q}_0$. Then, set $\mathcal{S}_1 = \mathcal{N}_1 = \{\mathcal{Q}_0\}$, $l = 1$ and $\gamma^0 = +\infty$.

Step 1) For each $\mathcal{Q} \in \mathcal{N}_l$, solve LMI problems described by equations (27) to (28) and by equations (29) to (**??**) to obtain $LB(\mathcal{Q})$ and $UB(\mathcal{Q})$. Find the minimal UB to update the current value γ^l and the current best solution $\mathbf{k}^l$.

Step 2) In $\mathcal{S}_l$, delete all $\mathcal{Q}$ such that $LB(\mathcal{Q}) - \gamma^l > -\epsilon LB(\mathcal{Q})$. Let $\mathcal{R}_l$ be the set of remaining hypercubes. If $\mathcal{R}_l = \emptyset$, terminate with γ^l the best ϵ-suboptimal value corresponding to the gain matrix $\mathbf{k}^l$.

Step 3) Choose $\mathcal{Q}_l \in argmin\{LB(\mathcal{Q}) | \mathcal{Q} \in \mathcal{R}_l\}$ and bisect it into two smaller hypercube $\mathcal{Q}_{l,1}$ and $\mathcal{Q}_{l,2}$. Let $\mathcal{N}_{l+1} = \{\mathcal{Q}_{l,1}, \mathcal{Q}_{l,2}\}$ and $\mathcal{S}_{l+1} = (\mathcal{R}_l \mathcal{Q}_l) \cup \mathcal{N}_{l+1}$. Set $l \leftarrow l + 1$ and go back to step 1).

Since $LB(\mathcal{Q})$ and $UB(\mathcal{Q})$ converge to the same value when the hyper-volume of the hypercube $\mathcal{Q}$ converge to zero, the algorithm presented in this subsection terminate after a finitely many iterations, yielding the ϵ-suboptimal value of problem described by equations (24), (20), (21), (23), (25) and (26).

6.APPLICATION EXAMPLE

The optimization approach described on the last section can be used to tune optimally the PID controller gains for arbitrary plant, sensors and reference model of Figure. 1. To illustrate this method, the reference model matrices and the plant model matrices have been chosen according to equations (1) and (2)

$$\left[\begin{array}{c|c} \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_r & \end{array}\right] = \left[\begin{array}{c|c} -0.4 & 1 \\ \hline 0.4 & \end{array}\right] \tag{31}$$

$$\left[\begin{array}{c|c} \mathbf{A}_p & \mathbf{B}_p \\ \hline \mathbf{C}_p & \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -4 & -6 & -4 & 1 \\ \hline 1 & 0 & 0 & 0 & \end{array}\right] \tag{32}$$

According to equations (3), the sensors model matrices has been chosen as: $\mathbf{A}_s = \mathbf{B}_s = \mathbf{C}_s = \emptyset$ and $\mathbf{D}_s = 1$. To match the output of the plant with the output of the reference model for a step input response, the weight error model matrices have been chosen according to equation (4) as

$$\left[\begin{array}{c|c}\mathbf{A}_w & \mathbf{B}_w \\ \hline \mathbf{C}_w & \end{array}\right] = \left[\begin{array}{c|c} -0.004 & 1 \\ \hline 1 & \end{array}\right] \quad (33)$$

Notes that this weight error model is a stable approximation of the unit step function. The impulse response of the complete system is then close to the step response of the system without weight error model. Finally, the controller model have been chosen to be a PID controller with a 100 rad/s low-pass filter. The matrices of this structured controller model are

$$\left[\begin{array}{c|c}\mathbf{A}_c & \mathbf{B}_c \\ \hline \mathbf{C}_c & \mathbf{D}_c \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -100 & 100 \\ \hline k_1 & k_2 - 100k_3 & 100k_3 \end{array}\right] \quad (34)$$

and the controller gains are given as $\mathbf{k} = \left[\begin{array}{ccc} k_i & k_p & k_d \end{array}\right]^T$ where k_i, k_p and k_d are respectively the integral, the proportional and the derivative controller gains. Algorithm 1 has been used to solve the optimization problem described by equations (24), (20), (21), (23), (25) and (26). The initial hypercube has been chosen as $\mathcal{Q}_0 = \{\mathbf{k}|0 < k_j < 10,\ j = 1...3\}$ and the tolerance has been chosen as $\epsilon = 0.25$. For the lower bound estimation problem, the ε has been chosen as 1×10^{-6}. BB Algorithm has been programmed by using the LMI MATLAB toolbox. The algorithm converged after 6128 iterations. The best upper bound obtained was UB = 0.049. This optimal solution corresponds to the gains matrix $k = [0.3955, 1.2256, 2.1582]^T$. The step response of the reference model and the feedback control system are shown in Figure 2.

7.CONCLUSION

In this paper, a BB algorithm has been used to designed a structured controller that minimizes a matching model criteria. The optimization problem has been formulated as a BMI problem with additional variables allowing lower and upper bound estimation by LMI subproblems. This optimization approach has been used to tune optimally a PID controller gains for arbitrary plant, sensors and reference model.

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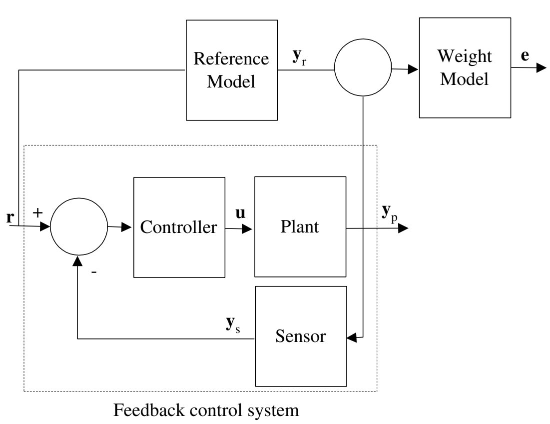

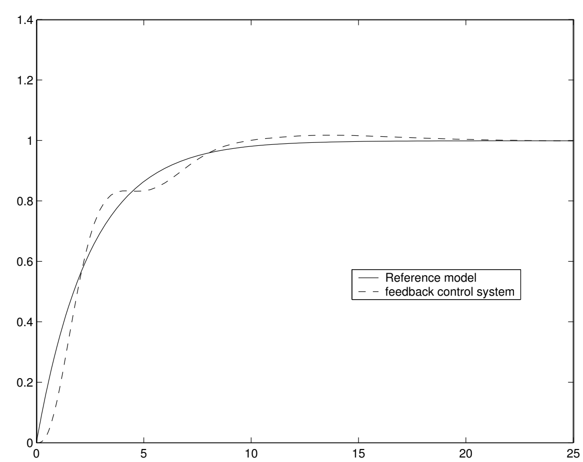

Figure 1: blocks-Diagram of the system. Figure2: Reference model and feedback control system step response.