STRUCTURED OPTIMAL CONTROLLER DESIGN USING BMI FORMULATION

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Abstract – In this paper, a BB algorithm is used to designed structured controller that minimizes a matching model criteria. This non convex optimization problem is formulated as a BMI problem with additional variables allowing lower and upper bound estimation by LMI subproblems. A reduction of the additional variables is proposed to reduce the computation time. To demonstrate the effectiveness of the approach, this global optimization method is used to tune optimally the PID controller gains.

Keywords: structured controller, non convex optimization, model matching, PID controller.

1.INTRODUCTION

Over the past decades, many advances have been made in the field of linear optimal control. In particular, several methods based on Linear Matrix Inequalities (LMI) formulation have been proposed to solve multiple optimal control problems for state feedback and full-order output feedback controllers [SCH, 97]. Recently, optimal performances of low-order output feedback and static output feedback controllers have been extensively studied. The underlying optimization problems are difficult to solve since they are non convex. LMI constraints with a non convex objective, described by a rank minimization, is a formulation of the low-order controller design problem. This particular problem can be transformed in a smooth concave optimization problem that can be solved by local optimization methods [GAH, 94] or by genetic algorithms [DU, 02]. A conservative solution of the low-order controller design problem by using a coprime factorization and positive real lemma expressed in LMI form has also been proposed in [WAN, 00], [WAN, 99] .

For static output feedback, fixed order controllers and structured controller, the design is also a non convex optimization problem. Approximative solution of the stabilization problem associated to this feedback control system can be found by using LMI formulation for a prescribed degree of stability [BEN, 98]. The optimal static output feedback and structured controller control problem can also be solved by using iterative local solution based on multiple LMI problem solutions [ROT, 94], [GER, 95], [GAH, 94]. Meanwhile the global minima can be found by using branch and bound (BB) algorithm [BAL, 92]. Also, the static output feedback and the structured control problem can be transformed into a Bilinear Matrix Inequality (BMI) optimization problem [SAF, 94], [LEE, 99]. The solution can then be solved by using local minimization approach [HAS, 99], [LEE, 99], [GOH, 94]. In this case, the global solution of the non convex optimization problem cannot be guaranteed. The branch and bound (BB) algorithm allows to find the global solution of the BMI problem [TUA, 00b] [TUA, 00a].

In this paper, a BB algorithm is used to designed a structured controller [LEE, 99] that minimizes a matching model criteria. This class of controller is very important for industrial application since its order and its structure can be imposed arbitrary. The optimization problem is formulated as a BMI problem with the additional variables that allow us to formulate the LMI subproblems associated to the lower and the upper bound estimations. Those estimations are necessary to formulate the associated BB algorithm. As in [TUA, 00a], the additional variables are the product of the decision variables. However, according to our specific problem, a matrix product formulation allows us to reduce the number of additional variables. This reduction allows to reduce the computation time. The transformation of constraints in the new variables space must then be different of the one proposed in [TUA, 00a] to take into a compte the variable reduction.

This paper is organized as follows. The system model is presented in section 2. Section 3 presents the optimization problem formulation. In section 4, the optimization approach is applied to tune optimally PID controller gains for arbitrary plant, sensors and reference model.

I. System model

Continuous linear invariant feedback control systems are generally characterized by a plant, that is the system to be controlled, sensors, and a feedback controller. In this paper, it is assumed that the controller is structured and has fixed order. The gains adjustment that minimizes the difference between a reference model and the closed loop of the feedback control system is then a non convex optimization problem.

The blocks-diagram corresponding to this complete system is illustrated by Fig. 1. As it can be seen, to be more flexible, the error between the reference model and the closed loop system is weighted by a weight error model.

Each block of the diagram in Fig. 1 is modelized in its state space form. The reference model is described as

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{A_r x_r} + \mathbf{B_r r} \\ \mathbf{y_r} &= \mathbf{C_r x_r} \end{aligned} \tag{1}$$

where $\mathbf{x_r} \in \mathbf{R}^{\mathbf{n_r}}$, $\mathbf{y_r} \in \mathbf{R}^{\mathbf{n_y}}$ and $\mathbf{r} \in \mathbf{R}^{\mathbf{n_y}}$. The plant model

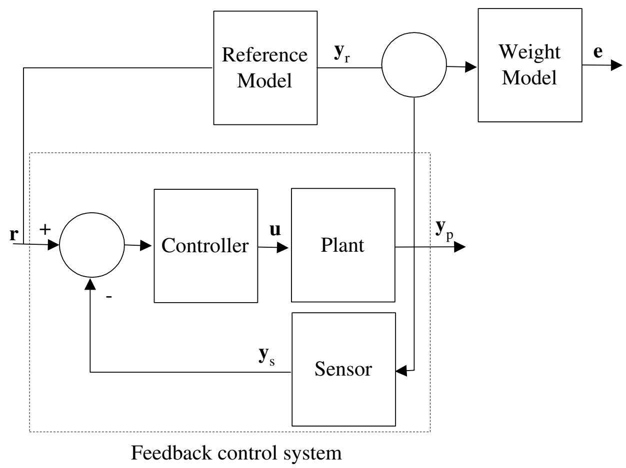

Fig. 1. blocks-Diagram of the system.

is described as

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{A_p x_p} + \mathbf{B_p u} \\ \mathbf{y_p} &= \mathbf{C_p x_p} \end{aligned} \tag{2}$$

where $\mathbf{x_p} \in \mathbf{R^{n_p}}$, $\mathbf{y_p} \in \mathbf{R^{n_y}}$ and $\mathbf{u} \in \mathbf{R^{n_u}}$. Then, the sensors model is described as

$$\begin{aligned} \dot{\mathbf{x}}_s &= \mathbf{A_s x_s} + \mathbf{B_s y_p} \\ \mathbf{y_s} &= \mathbf{C_s x_s} + \mathbf{D_s y_p} \end{aligned} \tag{3}$$

where $\mathbf{x_s} \in \mathbf{R^{n_s}}$ and $\mathbf{y_s} \in \mathbf{R^{n_y}}$. The weight error model is described as follows

$$\begin{aligned} \dot{\mathbf{x}}_w &= \mathbf{A_w x_w} + \mathbf{B_w}\varepsilon \\ \mathbf{e} &= \mathbf{C_w x_w} \end{aligned} \tag{4}$$

where $\varepsilon = \mathbf{y_r} - \mathbf{y_p} = \mathbf{C_r x_r} - \mathbf{C_p x_p}$, $\mathbf{x_w} \in \mathbf{R^{n_w}}$ and $\mathbf{e} \in \mathbf{R^{n_y}}$. Finally, the structured controller model is described as

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{A_c(k) x_c} + \mathbf{B_c(k)(r - y_s)} \\ \mathbf{u} &= \mathbf{C_c(k) x_c} + \mathbf{D_c(k)(r - y_s)} \end{aligned} \tag{5}$$

where $\mathbf{x_c} \in \mathbf{R^{n_c}}$ and [LEE, 99]

$$\begin{bmatrix} \mathbf{A_c(k)} & \mathbf{B_c(k)} \\ \mathbf{C_c(k)} & \mathbf{D_c(k)} \end{bmatrix} = \begin{bmatrix} \mathbf{A_{c0}} & \mathbf{B_{c0}} \\ \mathbf{C_{c0}} & \mathbf{D_{c0}} \end{bmatrix} + \mathbf{\Theta_L K_s(k) \Theta_R} \tag{6}$$

with

$$\mathbf{\Theta_L K_s(k) \Theta_R} = \sum_{\mathbf{i=1}}^{\mathbf{n_k}} \mathbf{\Theta_{Li} k_i \Theta_{Ri}} \tag{7}$$

where $\mathbf{k} \in \mathbf{R^{n_k}}$ is the gain vector, $\mathbf{\Theta_{Li}} \in \mathbf{R^{n_u \times n_{k_i}}}$ and $\mathbf{\Theta_{Ri}} \in \mathbf{R^{n_{k_i} \times n_s}}$ are full rank matrices. The system described by equations (1), (2), (4), (5) and (6) can be rewritten in one state space model as

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathcal{A} + \mathcal{B}_\mathbf{u} \mathbf{K_s(k)} \mathcal{C}_\mathbf{u})\mathbf{x} + (\mathcal{B}_\mathbf{r} + \mathcal{B}_\mathbf{u} \mathbf{K_s(k)} \mathcal{D}_\mathbf{u})\mathbf{r} \\ \mathbf{e} &= \mathcal{C}\mathbf{x} \end{aligned} \tag{8}$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_r} \\ \mathbf{x_c} \\ \mathbf{x_m} \\ \mathbf{x_w} \\ \mathbf{x_p} \end{bmatrix}, \quad \mathcal{B}_\mathbf{u} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_p} \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} \mathbf{A_r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{c0}} & -\mathbf{B_{c0} C_s} & \mathbf{0} & -\mathbf{B_{c0} D_s C_p} \\ \mathbf{0} & \mathbf{0} & \mathbf{A_s} & \mathbf{0} & \mathbf{B_s C_p} \\ \mathbf{B_w C_r} & \mathbf{0} & \mathbf{0} & \mathbf{A_w} & -\mathbf{B_w C_p} \\ \mathbf{0} & \mathbf{B_p C_{c0}} & -\mathbf{B_p D_{c0} C_s} & \mathbf{0} & \mathbf{A_p} - \mathbf{B_p D_{c0} D_s C_p} \end{bmatrix},$$

$$\mathcal{B}_\mathbf{r} = \begin{bmatrix} \mathbf{B_r} \\ \mathbf{B_{c0}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{B_p D_{c0}} \end{bmatrix}, \quad \mathcal{C}_\mathbf{u}^\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C_s^T D_c^T} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C_p^T D_s^T} \end{bmatrix},$$

and

$$\mathcal{C}^\mathbf{T} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{C_w^T} \\ \mathbf{0} \end{bmatrix} \quad \texttt{and} \ \mathcal{D}_\mathbf{u} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

II. Optimization approach

According to the closed loop system described by equation (8), the objective is to find the controller gain vector $\mathbf{k}$ to minimize the error $\mathbf{e}$. This problem can be formulated as

$$\mathbf{k} = \arg\min_{\mathbf{k}} \|\mathbf{H}_{\varepsilon\mathbf{r}}\|_\mathbf{2} \tag{9}$$

where

$$\mathbf{H}_{\varepsilon\mathbf{r}}\mathbf{(s)} = \mathcal{C}\left(\mathbf{sI} - (\mathcal{A} + \mathcal{B}_\mathbf{u}\mathbf{K_s(k)}\mathcal{C}_\mathbf{u})\right)^{-1}\left(\mathcal{B}_\mathbf{r} + \mathcal{B}_\mathbf{u}\mathbf{K_s(k)}\mathcal{D}_\mathbf{u}\right) \tag{10}$$

and $\|\mathbf{H}\|_2$ is the 2-norm of system $\mathbf{H}$ that exist only if $\mathbf{H}$ is Hurwitz-stable. if $\mathbf{H}$ is Hurwitz-stable, $\|\mathbf{H}\|_2$ is defined by [SCH, 97]

$$\|\mathbf{H}\|_2^2 = \frac{1}{2\pi} \ \texttt{trace} \int_{-\infty}^{\infty} \mathbf{H(j\omega)H(j\omega)^* d\omega} \tag{11}$$

To avoid the integral evaluation, the minimization problem described by equations (9) can be rewritten as [SCH, 97]

$$\mathbf{k} = \arg\min(\ \texttt{trace}(\mathcal{L}_\mathbf{1}(\mathcal{B}_\mathbf{r} + \mathcal{B}_\mathbf{u}\mathbf{K_s(k)}\mathcal{D}_\mathbf{u}, \mathbf{P}))) \tag{12}$$

where $\mathbf{P}$, a symmetric semi-positive definite matrix, is the solution of the following equations :

$$\mathcal{L}_2((\mathcal{A} + \mathcal{B}_\mathbf{u}\mathbf{K_s(k)}\mathcal{C}_\mathbf{u}), \mathbf{P}) + \mathbf{C^T C} = \mathbf{0}, \tag{13}$$

$$\mathcal{L}_1(\mathbf{B}, \mathbf{P}) = \mathbf{B^T P B} \tag{14}$$

and

$$\mathcal{L}_2(\mathbf{A}, \mathbf{P}) = \mathbf{PA} + \mathbf{A^T P} \tag{15}$$

Notice that in this minimization problem the H_2 norm could be substituted by H_∞ norm with minor modifications of the problem formulation.

Even though the problem described by equations (12) to (15) is non convex, BB algorithm can be used to find its global solution. The solution $\mathbf{k}$ is then searched in a hypercube $\mathcal{Q}$ [BAL, 92]. The BB formulation consists of finding a lower bound estimation and an upper bound estimation of the objective function when the decision variables are constrained in a hypercube $\mathcal{Q}$. The BB algorithm is based on the hypercube subdivision (branch) and on the evaluation of lower and upper bounds (bound) in the subdivided hypercubes $\mathcal{Q}_l$. The BB algorithm converges to the global optimal solution if the lower and the upper estimations converge to the same result while the hyper-volume of the subdivided hypercubes $\mathcal{Q}_l$ converges to zero. This condition ensures a global convergence of the algorithm. However, if the distance between the lower and the upper bound is very high until the hyper-volume of the hypercubes $\mathcal{Q}_l$ is very close to zero, the convergence is very slow. It is thus very important to find a lower and an upper bound estimations that are as close as possible to each other. Balakrishnan & Boyd [BAL, 92] propose to use the evaluation of the 2-norm at the center of the hypercube for the upper bound estimation. For the lower bound, they propose to use a loop transformation and the perturbation theory to obtain a conservative estimation. Unfortunately, BB algorithm with this lower and upper bound formulation converge very slowly since the lower bound estimation is too conservative.

It is the reason why in this paper, the minimization problem described by (12) to (15) will be transformed into a Bilinear Matrix Inequalities (BMI). Then, a BB algorithm will be used with an LMI minimization problem for the lower bound estimation. This approach has been presented in [TUA, 00b] and has been modified for different class of problems in [TUA, 00a].

A. BMI formulation

The minimization problem described by equations (12) to (15) can be transformed in a BMI formulation [SCH, 97]:

$$\mathbf{k} = \arg\min(\ \mathtt{trace}(\mathbf{Z})) \tag{16}$$

satisfying

$$\begin{gathered} \mathbf{P} > \mathbf{0} \\ \mathcal{L}_2((\mathcal{A} + \mathcal{B}_\mathbf{u}\mathbf{K_s}(\mathbf{k})\mathcal{C}_\mathbf{u}), \mathbf{P}) + \mathbf{C^T C} < \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & * \\ (\mathcal{B}_\mathbf{r} + \mathcal{B}_\mathbf{u}\mathbf{K_s}(\mathbf{k})\mathcal{D}_\mathbf{u})^\mathbf{T}\mathbf{P} & Z \end{bmatrix} > 0 \end{gathered} \tag{17}$$

where the symbol * is used to simplify the writing since matrices are symmetric. BMI formulation described by equations (16) and (17) cannot be transformed into the LMI form by using a change of variable since feedback controller of Figure. 1 is not a state feedback controller nor a full-order output feedback controller. As proposed in [TUA, 00b], BMI optimization problem described by equations (16) and (17) can be solved by using the BB algorithm. The key to this approach is to find LMI optimization problems to estimate an upper and a lower bound of BMI problem in an hypercube $\mathcal{Q}_l$. To accomplish this formulation, the BMI problem is first transformed according to an additional decision variables and the hypercube constraint is added. The additional variables are defined as [TUA, 00a]:

$$\mathbf{W_i} = \mathbf{k_i P}\mathcal{B}_\mathbf{u}^\mathbf{i},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k} \tag{18}$$

where k_i are the component of the gain vector $\mathbf{k}$ and, according to equation (6), $\mathcal{B}_\mathbf{u}^\mathbf{i}$ is defined as:

$$\mathcal{B}_\mathbf{u}^\mathbf{i} = \mathcal{B}_\mathbf{u}\mathbf{\Theta_{Li}}. \tag{19}$$

The additional variables described by equation (18) are different of the ones proposed in [TUA, 00a]. In fact, in [TUA, 00a], the different elements of matrix $\mathbf{P}$ are all considered as additional variables while in the additional variables described by equation (18), only the different elements of product $\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i}$ are considered. If we assumed that n_{k_i} is less than (n+1)/2 (where $n = n_r + n_p + n_s + n_w + n_c$), then, the number of additional variables defined by equation (18) is reduced from $n_k(n+1)n/2$ to $n\sum_{i=0}^{n_k} n_{k_i}$. Now, the additional variables must be incorporated in a new optimization problem formulation. The following proposition allows us to perform an appropriate transformation of the constraints to incorporate the additional variables.

Proposition 1 *If the hypercube $\mathcal{Q}_l$ is defined as follows*

$$\mathcal{Q}_l = \{\mathbf{k}|\underline{\mathbf{k}}_\mathbf{i}^\mathbf{l} < \mathbf{k_i} < \overline{\mathbf{k}}_\mathbf{i}^\mathbf{l},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k}\}, \tag{20}$$

$n_{k_i} \leq (n+1)/2$ and $\mathcal{B}_\mathbf{u}^\mathbf{i}$ are full rank matrix, Then, the following constraints set

$$\mathbf{W_i} = \mathbf{k_i P}\mathcal{B}_\mathbf{u}^\mathbf{i},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k} \tag{21}$$

$$\mathbf{P} > \mathbf{0} \tag{22}$$

$$\mathbf{k} \in \mathcal{Q}_\mathbf{l} \tag{23}$$

is equivalent to the constraints set described by equations (21), (22) and the following one

$$\begin{bmatrix} \tilde{k}_i^l \mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i} & * \\ \mathbf{W_i} - \hat{\mathbf{k}}_\mathbf{i}^\mathbf{l}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i} & \tilde{k}_i^l\mathbf{P} \end{bmatrix} > 0,\ i = 1...n_k \tag{24}$$

where $\tilde{k}_i^l = \frac{1}{2}(\overline{k}_i^l - \underline{k}_i^l)$ and $\hat{k}_i^l = \frac{1}{2}(\overline{k}_i^l + \underline{k}_i^l)$.

Proof: According to the constraint described by (22), the last constraints of the set described by equations (21), (22) and (23) can be rewritten as:

$$(k_i - \hat{k}_i^l)^2\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i} < \tilde{\mathbf{k}}_\mathbf{i}^\mathbf{l2}\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k} \tag{25}$$

and can be rewritten again as

$$(k_i - \hat{k}_i^l)\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i}(\mathbf{k_i} - \hat{\mathbf{k}}_\mathbf{i}^\mathbf{l}) < \tilde{\mathbf{k}}_\mathbf{i}^\mathbf{l2}\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k} \tag{26}$$

The constraints described by equation (21) is further used to transform the last constraint into the following form:

$$(\mathbf{W}_\mathbf{i}^\mathbf{T} - \hat{\mathbf{k}}_\mathbf{i}^\mathbf{l}\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P})\mathbf{P}^{-1}(\mathbf{W_i} - \hat{\mathbf{k}}_\mathbf{i}^\mathbf{l}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i}) < \tilde{\mathbf{k}}_\mathbf{i}^\mathbf{l2}\mathcal{B}_\mathbf{u}^\mathbf{iT}\mathbf{P}\mathcal{B}_\mathbf{u}^\mathbf{i},\ \mathbf{i} = \mathbf{1}...\mathbf{n_k} \tag{27}$$

The transformation of the constraint from the form given by equation (27) to that given by equation (24) is finally obtained by using the Schur complement.

Proposition 1 can then be used to transform the BMI problem described by equations (16), (17), (18) and (24) into the following equivalent form:

$$\mathbf{K} = \arg\min(\ \mathtt{trace}(\mathbf{Z})) \tag{28}$$

satisfying constraints given by equations (21), (22), (24) and

$$\mathcal{L}_2(\mathcal{A}, \mathbf{P}) + \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{n_k}} \mathcal{L}_\mathbf{2}(\mathbf{\Theta_{Ri}}\mathcal{C}_\mathbf{u}, \mathbf{W_i}) + \mathbf{C^T C} < \mathbf{0} \tag{29}$$

$$\begin{bmatrix} \mathbf{P} & * \\ (\mathbf{P}\mathcal{B}_\mathbf{r} + \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{n_k}} \mathbf{W_i}\mathbf{\Theta_{Ri}}\mathcal{D}_\mathbf{u})^\mathbf{T} & Z \end{bmatrix} > 0 \tag{30}$$

As in [BAL, 92], the upper bound can be found by considering the 2-norm at the center of the hypercube. By fixing the matrix gains at the center of the hypercube, the BMI described by equations (16) and (17) is transformed into LMI form:

$$UB(\mathcal{Q}_l) = \min(\ \mathtt{trace}(Z)) \tag{31}$$

satisfying

$$\begin{gathered}\mathbf{P} > \mathbf{0} \\ \mathcal{L}_2((\mathcal{A} + \mathcal{B}_{\mathbf{u}}\mathbf{K_s}(\hat{\mathbf{k}}^{\mathbf{l}})\mathcal{C}_{\mathbf{u}}), \mathbf{P}) + \mathbf{C^T C} < \mathbf{0} \\ \begin{bmatrix} \mathbf{P} & * \\ (\mathcal{B}_{\mathbf{r}} + \mathcal{B}_{\mathbf{u}}\mathbf{K_s}(\hat{\mathbf{k}}^{\mathbf{l}})\mathcal{D}_{\mathbf{u}})^{\mathbf{T}}\mathbf{P} & Z \end{bmatrix} > 0\end{gathered} \tag{32}$$

where $\hat{\mathbf{k}}^l$ is the gain vector $\mathbf{k}$ fixed at the center of hypercube $\mathcal{Q}_l$. For the lower bound estimation, the non convex constraints described by equations (21), (22), (24), (29) and (30) is transformed to a convex one by simply removed the equality constraint (21):

$$LB(\mathcal{Q}_l) = \min(\ \mathtt{trace}(Z)) \tag{33}$$

satisfying constraints given by equations (22), (24), (29), (30) and

$$\mathcal{L}_2(\mathcal{A}, \mathbf{P}) + \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{n_k}} \mathcal{L}_{\mathbf{2}}(\mathbf{\Theta_{Ri}}\mathcal{C}_{\mathbf{u}}, \mathbf{W_i}) + \mathbf{C^T C} > -\xi \mathbf{I} \tag{34}$$

where ξ is a small numerical value. Notice that the constraint described by equation (34) has been added to allow a better estimation of the lower bound. This constraint restricts the solution of the lower bound estimation problem but does not change the original BMI problem as described by equations (28), (21), (22), (24), (29) and (30).

B. Branch and bound algorithm

According to upper and lower bound estimation given by equations (31) to (32) and (33), (22), (24), (29), (30) and (34) , the BB algorithm allowing the solution of the BMI problem can be stated as follows [TUA, 00a]

Algorithm 1:

Step 0) Start with a tolerance factor ϵ and the initial hypercube $\mathcal{Q}_0$. Then, set $\mathcal{S}_1 = \mathcal{N}_1 = \{\mathcal{Q}_0\}$, $l = 1$ and $\gamma^0 = +\infty$.
Step 1) For each $\mathcal{Q} \in \mathcal{N}_l$, solve LMI problems described by equations (31) to (32) and by equations (33), (22), (24), (29), (30) and (34) to obtain $LB(\mathcal{Q})$ and $UB(\mathcal{Q})$. Find the minimal UB to update the current value γ^l and the current best solution $\mathbf{k^l}$.
Step 2) In $\mathcal{S}_l$, delete all $\mathcal{Q}$ such that $LB(\mathcal{Q}) - \gamma^l > -\epsilon LB(\mathcal{Q})$. Let $\mathcal{R}_l$ be the set of remaining hypercubes. If $\mathcal{R}_l = \emptyset$, terminate with γ^l the best ϵ-suboptimal value corresponding to the gain matrix $\mathbf{k^l}$.
Step 3) Choose $\mathcal{Q}_l \in argmin\{LB(\mathcal{Q}) | \mathcal{Q} \in \mathcal{R}_l\}$ and bisect it into two smaller hypercube $\mathcal{Q}_{l,1}$ and $\mathcal{Q}_{l,2}$. Let $\mathcal{N}_{l+1} = \{\mathcal{Q}_{l,1}, \mathcal{Q}_{l,2}\}$ and $\mathcal{S}_{l+1} = (\mathcal{R}_l \mathcal{Q}_l) \cup \mathcal{N}_{l+1}$. Set $l \leftarrow l + 1$ and go back to step 1).

Since $LB(\mathcal{Q})$ and $UB(\mathcal{Q})$ converge to the same value when the hyper-volume of the hypercube $\mathcal{Q}$ converge to zero, the algorithm presented in this subsection terminate after a finitely many iterations, yielding the ϵ-suboptimal value of problem described by equations (28), (21), (22), (24), (29) and (30).

III. Application example

The optimization approach described on the last section can be used to tune optimally the PID controller gains for arbitrary plant, sensors and reference model of Fig. 1. The specific example that has been chosen to show the good performances of the approach is illustrated by Fig. 2.

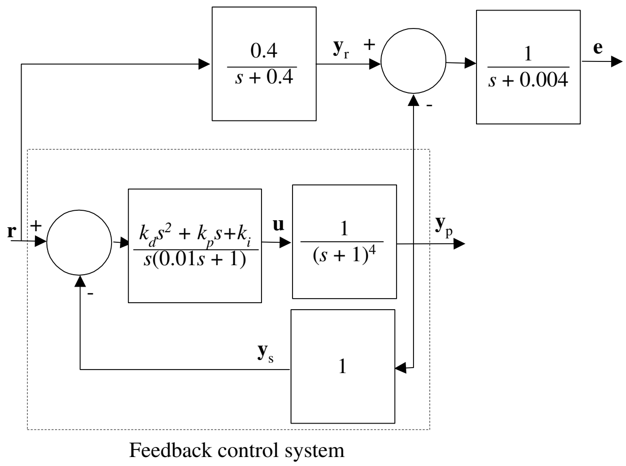

Fig. 2. Application example.

For this example, the reference model matrices, the plant model matrices, and the sensors model matrices are given according to equations (1), (2) and (3) as

$$\left[\begin{array}{c|c} \mathbf{A_r} & \mathbf{B_r} \\ \hline \mathbf{C_r} & \end{array}\right] = \left[\begin{array}{c|c} -0.4 & 1 \\ \hline 0.4 & \end{array}\right] \tag{35}$$

$$\left[\begin{array}{c|c} \mathbf{A_p} & \mathbf{B_p} \\ \hline \mathbf{C_p} & \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -4 & -6 & -4 & 1 \\ \hline 1 & 0 & 0 & 0 & \end{array}\right] \tag{36}$$

$$\left[\begin{array}{c|c} \mathbf{A_s} & \mathbf{B_s} \\ \hline \mathbf{C_s} & \mathbf{D_s} \end{array}\right] = \left[\begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & 1 \end{array}\right] \tag{37}$$

To match the output of the plant with the output of the reference model for a step input response, the weight error model matrices have been chosen according to equation (4) as

$$\left[\begin{array}{c|c} \mathbf{A_w} & \mathbf{B_w} \\ \hline \mathbf{C_w} & \end{array}\right] = \left[\begin{array}{c|c} -0.004 & 1 \\ \hline 1 & \end{array}\right] \tag{38}$$

Notes that this weight error model is a stable approximation of the unit step function. The impulse response of the complete system is then close to the step response of the system without weight error model. Finally, the controller model have been chosen to be a PID controller with a 100 rad/s low-pass filter. The matrices of this structured controller model are

$$\left[\begin{array}{c|c}\mathbf{A_c} & \mathbf{B_c}\\ \hline \mathbf{C_c} & \mathbf{D_c}\end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0\\ 0 & -100 & 100\\ \hline 0 & 0 & 0\end{array}\right] + k_1\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + k_2\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} + k_3\begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}0 & -100 & 100\end{bmatrix} \tag{39}$$

and the controller gains are given as

$$\mathbf{k} = \begin{bmatrix} k_i & k_p & k_d \end{bmatrix}^{\mathbf{T}} \tag{40}$$

where k_i, k_p and k_d are respectively the integral, the proportional and the derivative controller gains.

Algorithm 1 has been used to solve the optimization problem described by equations (28), (21), (22), (24), (29) and (30). The initial hypercube has been chosen as $\mathcal{Q}_0 = \{\mathbf{k}|\mathbf{0} < \mathbf{k_j} < \mathbf{10},\ \mathbf{j} = \mathbf{1}...\mathbf{3}\}$ and the tolerance has been chosen as $\epsilon = 0.25$. For the lower bound estimation problem, the ε has been chosen as 1×10^{-6}. BB Algorithm has been programmed by using the LMI MATLAB toolbox. the algorithm converged after 6128 iterations. The best upper bound obtained was UB $= 0.049$. This optimal solution corresponds to the gains matrix $k = [0.3955, 1.2256, 2.1582]^T$. The step response of the reference model and the feedback control system are shown in Fig. 3, while the impulse response of the system described by Fig. 1 is shown in Fig. 4.

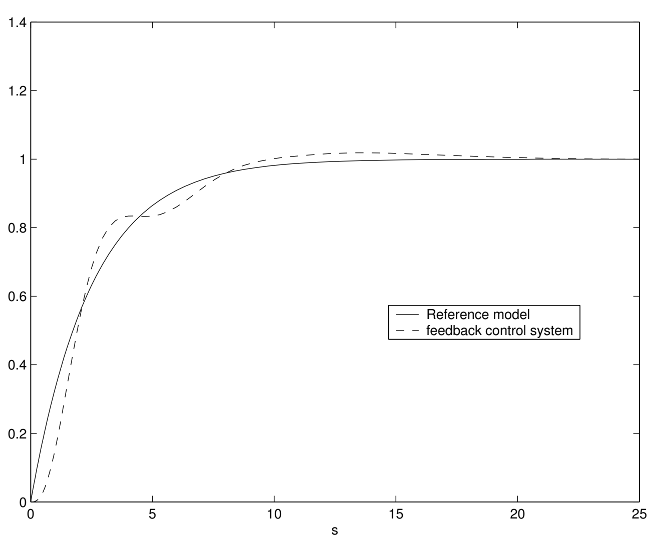

Fig. 3. Reference model and feedback control system step response.

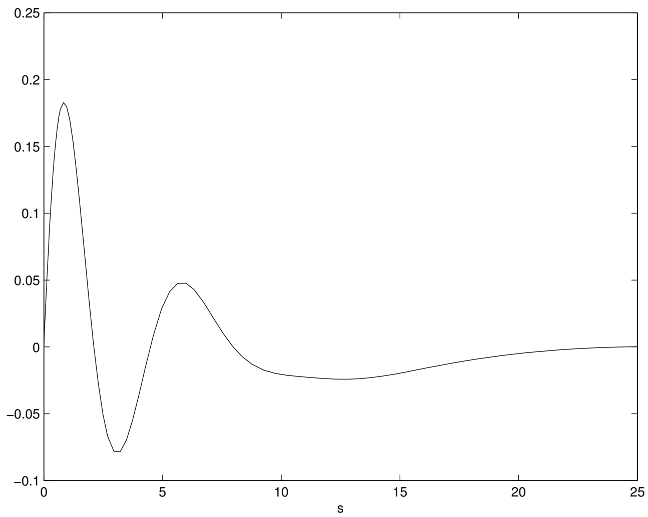

Fig. 4. Impulse response of the system illustrated by Fig. 1.

IV. Conclusion

In this paper, a BB algorithm has been used to designed a structured controller that minimizes a matching model criteria. The optimization problem has been formulated as a BMI problem with additional variables allowing lower and upper bound estimation by LMI subproblems. This optimization approach has been used to tune optimally a PID controller gains for arbitrary plant, sensors and reference model. For the future works, the robustness of the closed loop uncertain system will be incorporated in the problem formulation by using additional H_∞ constraint.

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