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Enhanced and restored signals as a generalized solution for shock filter models. Part I—existence and uniqueness result of the Cauchy problem

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Abstract

Signal enhancement and restoration is one of the fields that make extensive use of PDE theory. More specifically, some authors have proposed successive improved shock filters based on nonlinear hyperbolic equations. These models yield satisfactory results; however, a wider range of degrees of freedom when handling the model parameters (coefficients and components) would be of great interest because it would increase the model's efficiency and facilitate adaptation to specific situations. Naturally, the key challenge in proceeding thus is to ensure that the problem remains well-posed. In this paper, we propose a more general shock filter that introduces new parameters to control the shock speed. Interpreting the proposed model in a framework of generalized functions algebra, we prove existence and uniqueness solution results.

Keywords: Generalized functions; Partial differential equations (PDEs); Cauchy problem; Shock filters; Numerical schemes; Signals enhancement and restoration

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1. Introduction

Significant interest in the mathematical formulation of problems related to the field of image processing has been observed over the past decade. A large number of scientists have turned their attention to the physical, geometrical, and statistical models of the image content, the use of powerful optimization techniques, the development of variational approaches, non-linear methods, the intensive use of formal computation, the increased use of differential and projective geometry, algebraic tools, and the differential and semi-differential algebraic invariant theory in pattern analysis and recognition.

The majority of these techniques require using partial differential equations (PDEs). Image enhancement and restoration is one of the fields that make extensive use of PDEs. The approaches that were developed initially used a least square criterion (Wiener filter [3]) with algebraic methods in solution seek problems. Furthermore, a linear quadratic minimization criterion with constraints is proposed in [11], and APM (a posteriori maximum) methods are proposed in [9]. However, the limits of these approaches are quickly reached through the observed oscillating effects and the degradation results near the discontinuities. Consequently, new approaches that address the problem of noisy image enhancement and restoration as a smoothing issue have been proposed. A classical approach consists in using a convolution (smoothing) operation to reduce noise. The operator most frequently used is the Gaussian operator. Koendrink [12] shows that the Gaussian, used for convolution operations, can be rewritten as a diffusion process formulated by a parabolic PDE. This PDE, known as a hit equation, provides an isotropic diffusion. In this way, the process explicitly and considerably reduces the noise in the homogeneous intensity areas; however, the smoothing operation leads to the loss of pertinent information, thereby causing a degradation in the visual quality of the resulting image in areas where grey-level intensity discontinuities occur. Various ideas on anisotropic diffusion with a view to avoiding such information loss have surfaced. Perona and Malik [16] are the first to propose such a model. However, the model they present involves two major limitations. The first is its ineffectiveness in areas where the noise presents strong discontinuities. The second is of a theoretical nature related to the existence and uniqueness of a solution. In order to circumvent these shortcomings, an initial improvement using a smoothing version of the equation was proposed by Catte et al. in [5] and then by Alvarez et al. in [1]. Rudin first proposed the concept of shock filters for one-dimensional signal enhancement and restoration in [18]. This model was then improved by Osher and Rudin in [15]. Alvarez and Mazorra followed suit in [2] by proposing a new class of shock filters generalized to the two-dimensional case using the concept of directional derivatives, deriving parallel to the gradient direction to enhance the discontinuities, and perpendicular to the gradient direction for the smoothing operation.

In this paper, we begin by proposing a generalized model in the one-dimensional case of that proposed by Osher and Rudin [15]. The main motivation behind this generalization is to exercise tighter control of the enhancement and restoration process. Naturally, the greatest challenge is to keep the obtained model well-posed. This is clearly not obvious, since the coefficients used in the model are often discontinuous functions, and the processed signals are also discontinuous, thus their space derivatives are Dirac functions. We then deal with a product of distribution, which is non-sensical according to the classical theory of distribution (refer to [19] for the Schwartz theorem). In the framework of generalized functions, introduced by Colombeau [6], the product of distribution makes sense without contradicting the Schwartz theorem. This is made possible thanks to the introduction of the association notion, which is a generalization of the equality notion. An overview of this generalized functions theory is provided in this paper (a full explanation is found in [6]). This theory allows us to provide some interesting and necessary results when modeling with PDEs: among them, the existence and uniqueness of solutions (which will be proved in this paper) including the case of discontinued coefficients. We will also see that the smoothing properties introduced by Alvarez and Mazorra [2], which they show to markedly improve the results, are inherent to the model when interpreted in the generalized functions algebra; the smoothing aspect is therefore guaranteed. The proposed one-dimensional model is then generalized to two-dimensional signals (images). In particular, a maximum principle will be proved, which is crucial in the case of images because it causes the values of the produced images to remain between the minimal and maximal values of the original image at all times. In Section 2, we review previous works. In Section 3, an overview of the generalized functions algebra is given. In Sections 4 and 5, we propose our one- and two-dimensional models and prove theorems of existence and uniqueness of generalized solutions, as well as some theoretical discussions. We conclude in Section 6.

2. Overview of previous works

Rudin in [18] was the first to apply the concepts and techniques of the non-linear hyperbolic equation field to image enhancement. He proposed the following model:

$$u_t + F(u_{xx})|u_x| = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+,$$

where $F(\cdot)$ is a function such that $F(s)s \geqslant 0$. To discretize this equation, an explicit monotone scheme is used, thereby preserving the total variation. As noted in [2], this scheme cannot remove certain kinds of noise, such as "salt and pepper" noise. Rudin's model generates a great number of spurious shocks at the Laplacian zero-crossings due to the influence of noise. In order to avoid these spurious shocks, Alvarez and Mazorra [2] proposed the following improved hyperbolic partial differential model, which follows the classical theory of Marr [13]:

$$u_t + F(G_\sigma * u_{xx}, G_\sigma * u_x)u_x = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+,$$

where $G_\sigma(\cdot)$ is a smoothing kernel, and F satisfies the condition cited above. Alvarez and Mazorra developed an interesting, implicit, unconditionally-stable scheme. First developed by the authors in [1], this model is generalized to the case of two-dimensional signals in keeping with the directional smoothing ideas mentioned in Section 1. They propose the following parabolic–hyperbolic equation:

$$u_t = CL(u) - u_\eta F(G_\sigma * u_{\eta\eta}, G_\sigma * u_\eta) \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+,$$

where $\eta = \eta(x, y)$ represents the direction perpendicular to the gradient $\nabla u(x, y)$, C is any positive constant, and $L(u)$ is any directional smoothing operator. This model yields satisfactory results, as shown in [2]. However, to give a larger scope of application to the models derived from a shock filter theory, we believe it is necessary to maintain heightened control of the created shocks' velocity. In the models cited previously, the velocity represented by the function $F(G_\sigma * u_{\eta\eta}, G_\sigma * u_\eta)$ controls the position where the model develops shocks; however, it controls neither the intensity of the velocity according to the features of the signal (image) in different regions, nor what we target in the enhancement and restoration process. In Section 3, we propose a model which takes these issues into account. The greatest challenge in doing so is to preserve the well-posedness of the obtained models. Before developing the proposed model, we shall give a brief overview of the generalized functions space, which is the framework in which this model is considered and in which the existence and uniqueness result is proven.

3. Overview of the generalized functions space

The following is an overview of the generalized functions algebra, also known as the Colombeau algebra. For details about this theory, see [3].

First, let A_q, $q = 0, 1, 2, \ldots,$ be the set defined by

$$A_q = \left\{ \varphi \in \mathcal{D}(\Re^n) \;\Big|\; \int \varphi \, dx = 1,\ \int x^j \varphi \, dx = 0 \text{ for } 1 \leqslant |j| \leqslant q \right\}.$$

We define by $\mathcal{E}(\Omega)$ the set of functions

$$R : A_1 \times \Omega \to \mathbb{R}, \quad (\Phi, x) \to R(\Phi, x),$$

where R is a $\mathcal{C}^\infty$ function on x for each fixed Φ. $\mathcal{E}(\Omega)$ is an algebra and $\mathcal{C}^\infty(\Omega)$ is a sub-algebra of $\mathcal{E}(\Omega)$.

Now define by $\mathcal{E}_M(\Omega)$ the sub-algebra of $\mathcal{E}(\Omega)$ functions that have a moderate increase. This means that

$$\begin{aligned} \mathcal{E}_M(\Omega) = \Big\{ & R \in \mathcal{E}(\Omega) \;\big|\; \forall K \text{ compact of } \mathbb{R}^n,\ \forall D \text{ derivative operator},\ \exists N \in \mathbb{N} \\ & \text{such that if } \Phi \in A_N,\ \exists \eta > 0,\ C > 0,\ \sup_{x \in K} \big| D R_\varepsilon(\Phi, x) \big| \leqslant C (1/\varepsilon)^N \\ & \text{with } 0 < \varepsilon < \eta < 1 \Big\}, \end{aligned}$$

where $R_\varepsilon(\Phi, x) = R(\Phi_\varepsilon, x)$ and $\Phi_\varepsilon(\lambda) = (1/\varepsilon^n)\Phi(\lambda/\varepsilon)$.

Let $\mathcal{I}(\Omega)$ be a subset of $\mathcal{E}_M(\Omega)$ defined by

$$\begin{aligned} \mathcal{I}(\Omega) = \Big\{ & R \in \mathcal{E}_M(\Omega) \;\big|\; \forall K \text{ compact of } \Omega,\ \forall D \text{ derivative operator},\ \exists N \in \mathbb{N} \\ & \text{such that } \forall \Phi \in A_N,\ \exists C > 0 \text{ and } \eta > 0,\ \forall q \text{ sufficiently large:} \\ & \sup_{x \in K} \big| D R_\varepsilon(\Phi, x) \big| \leqslant C (1/\varepsilon)^{q-N} \text{ with } 0 < \varepsilon < 0 \Big\}. \end{aligned}$$

$\mathcal{I}(\Omega)$ is an ideal of $\mathcal{E}_M(\Omega)$.

The $\mathcal{C}^\infty$ generalized functions algebra $\mathcal{G}(\Omega)$ is defined as the quotient

$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_M(\Omega)}{\mathcal{I}(\Omega)}.$$

This definition of the generalized functions algebra is rather abstract and, practically speaking, how to handle the elements of $\mathcal{G}(\Omega)$ is not apparent. Thus, we define a simplified space of the generalized functions algebra ($\mathcal{G}_s$) without a canonical inclusion of the distributions. Refer to [3, Chapter 8] regarding this immediate simplification.

If Ω is any open set in $\mathbb{R}^n$, we define the space of "simplified global generalized functions" as follows. The reservoir of representatives is

$$\begin{aligned}\mathcal{E}_s(\Omega) = \big\{&\text{all maps } R \in \mathcal{C}^\infty\big(]0,1] \times \Omega, \mathbb{R}\big), \text{ such that } \forall D\\ &\text{(partial } x\text{-derivative, including the identity) } \exists N \in \mathbb{N},\ c > 0,\\ &\text{such that } \forall x \in \Omega, \big|(DR)(\varepsilon, x)\big| \leqslant c/\varepsilon^N\big\},\end{aligned}$$

and the ideal of $\mathcal{E}_s(\Omega)$ is

$$\begin{aligned}\mathcal{N}_s(\Omega) = \big\{&R \in \mathcal{C}^\infty(\Omega), \text{ such that } \forall D,\ \forall q \in \mathbb{N}\ \exists c_q > 0,\\ &\text{such that } \forall x \in \Omega,\ \big|(DR)(\varepsilon, x)\big| \leqslant c_q \varepsilon^q\big\}.\end{aligned}$$

Then the space $\mathcal{G}_{s,g}(\Omega)$ of the simplified global generalized functions on Ω is the quotient algebra

$$\mathcal{G}_{s,g}(\Omega) = \frac{\mathcal{E}_s(\Omega)}{\mathcal{N}_s(\Omega)}.$$

The term "global" is employed in this instance because the above bounds hold globally on Ω, and not only on compact subsets of Ω, as is the case in [3].

Recall now the definition of some useful operators in $\mathcal{G}_{s,g}(\Omega)$.

Definition 3.1 (The association concept). We say that $G_1, G_2 \in \mathcal{G}_s(\Omega)$ are associated (we write $G_1 \approx G_2$) if and only if for any Ψ in $\mathcal{D}(\Omega)$ we have

$$\int_\Omega \big[R_1(\varepsilon, x) - R_2(\varepsilon, x)\Psi(x)\big]\,dx \to 0 \quad \text{when } \varepsilon \to 0,$$

where $R_1 \in \mathcal{E}_s(\Omega)$ is a representative of G_1 and $R_2 \in \mathcal{E}_s(\Omega)$ is a representative of G_2.

The association could be viewed as a weak generalization of the equality concept. Consequently, when dealing with PDE equations, the equality is replaced by the association unless the equality is certain.

Definition 3.2 (The product of generalized functions). The product of two generalized functions is defined naturally by the class of the product of their representatives (by replacing the equality with the association concept), i.e., if $G_1, G_2 \in \mathcal{G}_s(\Omega)$ and R_1, R_2 are their respective representatives, then $G_1 \cdot G_2 \approx \text{class}\{R_1 \cdot R_2\}$. A non-linear regular function of generalized functions is defined in more general terms in [4].

Definition 3.3 (Derivatives of generalized functions). A derivative of a generalized functions ∂g is naturally defined by the class of equivalence of a given representative. That is, if $R(\varepsilon, x)$ is a representative of g, $\partial g = \text{class}\,\partial R(\varepsilon, x)$.

Definition 3.4 (Regularized derivatives). The concept of regularized derivatives $\bar{\partial}$ constitutes the basic ingredient ensuring existence and uniqueness results in many situations. The properties of these derivatives are discussed in [7,8]; we recall, for instance, that if g is a generalized functions then $\bar{\partial}g$ and ∂g are associated in $\mathcal{G}_{s,g}$. We shall now give the definition of $\bar{\partial}$. If $R(\varepsilon, x)$ is a representative of a generalized functions g, ∂ is a partial x-derivative, $\rho : \mathbb{R}^n \to \mathbb{R}$ with $\int \rho(\mu)\, d\mu = 1$ ($\rho \in \mathcal{D}(\Re^n)$, or ρ step function) is a "mollifier", and $h : \varepsilon \to h(\varepsilon)$ is a scaling function ($h : [0,1] \to [0,1]$ and $h(\varepsilon) \to 0$ when $\varepsilon \to 0$), then the regularized derivative $\bar{\partial}g$ is the class of $(\partial R(\varepsilon, \cdot) * \rho_{h(\varepsilon)})(x)$.

Now we will examine which classical functions or distributions can be represented by generalized functions.

Let f be a function in the space $\mathcal{D}_{L^\infty}(\Omega)$, i.e., f is a C^∞ function on Ω, globally bounded on Ω as well as its derivatives; then with f, associate $R(\varepsilon, x) = f(x)$. This gives the inclusion $\mathcal{D}_{L^\infty}(\Omega) \subset \mathcal{G}_s(\Omega)$. Let f be a function in the space $L^\infty(\mathbb{R}^n)$; then with f, associate $R(\varepsilon, x) = f * \rho_\varepsilon(x)$ with a chosen $\rho \in \mathcal{D}(\mathbb{R}^n)$, $\int \rho(\lambda)\, d\lambda = 1$ and $\rho_\varepsilon(\lambda) = (1/\varepsilon^n)\rho(\lambda/\varepsilon)$. For any given mollifier ρ, this gives an inclusion $L^\infty(\mathbb{R}^n) \subset \mathcal{G}_s(\mathbb{R}^n)$. In more general terms, let T be a distribution in $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$, i.e., T is a finite sum of the derivatives of functions in $L^\infty(\mathbb{R}^n)$. With T, associate $R(\varepsilon, x) = (T * \rho_\varepsilon)(x)$ as above. Therefore, for a given ρ, this provides an inclusion of $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ in $\mathcal{G}_s(\mathbb{R}^n)$. Similarly, there is an inclusion of $\mathcal{E}'(\Omega)$—the space of all distributions with compact support—in $\mathcal{G}_s(\Omega)$. All these inclusions become canonical, i.e., the arbitrariness in the choice of a mollifier ρ disappears if we work in the space $\mathcal{G}(\Omega)$ of "non-simplified" generalized functions, exactly as in [6, Chapter 8], whose definition is slightly more complex.

Finally, let us give the following proposition that plays an important role in the definition of the sense of convergence of some numerical schemes for instance.

Proposition 3.1. *Let $T \in \mathcal{D}'(\Omega)$. Then there exists $g \in \mathcal{G}_{s,g}(\Omega)$ associated with T. We write $g \approx T$ and say that T is the "macroscopic aspect" of g.*

The proof is given in [3,4].

4. The one-dimensional proposed model

In this section, we propose a generalized Rudin shock filter model in one-dimensional case. An interpretation of the model in the generalized functions algebra framework is developed, and then existence and uniqueness results are proved. A generalization of the model to a two-dimensional case with an extension of the theoretical results established for the one-dimensional case is proposed in Section 5.

In light of the motivations raised in Sections 1 and 2, we propose the following one-dimensional quasi-linear equation with discontinuous coefficients as a shock filter model:

$$u_t + a(x)F(u_{x^2}, u_x)\partial_x f\big(u(x)\big) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+. \tag{1}$$

The function a is a bounded and measurable function. F and f are regular (smooth) functions. The convolution of F arguments is omitted here. We shall see that the smoothing operation is inherent to the model when the problem is seen over the generalized functions theory. The function F plays the same role as in the previous models, meaning that it controls the positions where the model must develop shocks then enhance the contours. The introduction of the coefficients $a(x)$ and f control the shock velocity according to characteristics of the original signal and/or what is targeted. They can, for instance, focus processing on specific regions by setting the function equal to the characteristic function of the target region (this is a specific case where $a(x)$ is a discontinuous function). The coefficient $a(x)$ can also serve to achieve better control of contour detection using the well-known filters commonly employed for this task. With this coefficient, we can also simplify the model by setting the function F equal to one, and having a play the same role as F, but using the initial condition (signal), namely $F \equiv 1$ and $a(x) = G(u^0_{xx}, u^0_x)$, where G is a shock detection function. This function is therefore only computed once, at the beginning of the process. Furthermore, by doing so, we believe we minimize the impact of the problem of the edges' location, which may shift during the process. Note that this choice does not make sense from the classical distribution theory point of view. In Part II of this paper (submitted separately), we demonstrate through testing how controlling the propagation speed (velocity) can render the restoration process appreciably faster. Lastly, the function f also allows the shocks' velocity to be controlled, but only according to the signal produced each time.

An interpretation of a general Cauchy problem within the framework of the generalized functions theory is proposed in [7,14]. This interpretation consists of a regularization of space derivatives. An adequate choice of mollifiers leads to a discretization in the direction of the information (i.e., in the direction from which the characteristic curves originate). It is proven that the macroscopic aspect of the generalized solution is the classical entropic solution for the scalar equation $\partial_t u(x,t) + \partial_x f(u(x,t)) = 0$. A generalization of this study to the problem $\partial_t u(x,t) + a(x)\partial_x f(u(x,t)) = 0$, where $a(x)$ is a discontinuous bounded function, is developed in [4,17]. Along the same lines as in [7,17], we will study problem (1) and show that the formulation of this equation in G appears as a viscosity method. The diffusion aspect (noise elimination) therefore occurs as a natural process.

4.1. Existence and uniqueness of a generalized solution

Constructing a generalized solution to problem (2) now becomes a classical process (see [7,8,10,14]). Problem (1) is rewritten with the regularized derivatives in space and the usual derivatives in time are retained. We replace u^0 and a by associated generalized functions. In another words, u^0, a are replaced by their images in $\mathcal{G}_{s,g}$ using the embedding of L^∞ into $\mathcal{G}_{s,g}$. Then, we prove the existence and uniqueness solution for some representatives of u^0 and a images, and that the regular solution obtained is in $\mathcal{E}_{M,s}$. Hence its class is a solution of (2). Finally we prove the uniqueness of the solution as an element of $\mathcal{G}_{s,g}$.

Let us replace the coefficient a and the initial condition u^0 by an associated generalized functions A and U^0. Representatives of such functions are obtained by a regularization of a and u^0 through a convolution process. More specifically, we consider the following equivalent classes:

$$A = \text{class}\big\{a^{h_1(\varepsilon)}(\cdot),\ 0 < \varepsilon < 1\big\}, \qquad U^0 = \text{class}\big\{u^{0,h_3(\varepsilon)}(\cdot),\ 0 < \varepsilon < 1\big\},$$

where

$$a^{h_1(\varepsilon)} = a * \left(\frac{1}{h_1(\varepsilon)}\rho\left(\frac{x}{h_1(\varepsilon)}\right)\right), \qquad u^{0,h_2(\varepsilon)} = u^0 * \left(\frac{1}{h_2(\varepsilon)}\rho_1\left(\frac{x}{h_2(\varepsilon)}\right)\right),$$

ρ and ρ_1 are C^∞ smoothing functions (mollifiers) supported on the unit ball. The operator $*$ is the convolution product. The scale variables $h_1(\varepsilon)$ and $h_2(\varepsilon)$ are functions which tend conveniently to 0, and are chosen so that the previous functions and their derivatives are moderate functions. To do so, choosing $h_i(\varepsilon) = (\log(1/\varepsilon))^{-1}$ is sufficient in order to have $\exp(h_i(\varepsilon)) = O(1/\varepsilon^N)$, for $N \in \mathbb{N}$.

Note: in the following, we refer to $a^{h_1(\varepsilon)}$ by a^ε and $u^{0,h_3(\varepsilon)}$ by $u^{0,\varepsilon}$ in order to simplify the notation.

We finally deal with the equation

$$\begin{aligned} &U_t + A(x)F\big(\bar\partial_{x^2}U, \bar\partial_x U\big)\bar\partial_x f\big(U(x)\big) = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^+,\\ &U(0,x) = U^0. \end{aligned} \tag{2}$$

Consider Eq. (2) in its equivalent form

$$U_t + A(x)F\big(\bar\partial_{x^2}U, \bar\partial_x U\big)f'(U)\bar\partial_x U(x) = 0.$$

As we mentioned above, to prove the existence of the generalized function U, we start by proving the existence of some representative u^ε satisfying (the regularized derivatives are replaced by their convolution expression using mollifiers ρ and ρ^1):

$$\begin{aligned} &u_t^\varepsilon + a^\varepsilon(x)F\big(\partial_x\big[\partial_x[u^\varepsilon * \rho_{h(\varepsilon)}]\big] * \rho^1_{h(\varepsilon)}, \partial_x[u^\varepsilon * \rho_{h(\varepsilon)}](x,t)\big)\\ &\quad \times f'(u^\varepsilon)\partial_x[u^\varepsilon * \rho_{h(\varepsilon)}](x,t)(x) = 0. \end{aligned}$$

First, let us assume that $a(\cdot)F(\cdot,\cdot)f'(\cdot) > 0$. Assuming values greater than zero implies that the characteristic curves arriving at a given point have a positive slope, meaning that this point's cone of dependence is oriented to the left. This suggests that we choose the mollifier in the regularized x-derivative

$$\partial_x[u^\varepsilon * \rho_{h(\varepsilon)}](x,t) = \partial_x \int u^\varepsilon(x-y,t)\rho_{h(\varepsilon)}(y)\,dy,$$

such that this regularized derivative is taken on the left, i.e., $\rho(y) = 0$ if $y < 0$. For this study, we consider the simplified case $\rho(y) = 0$ if $y < 0$ and $\rho(y) = 1$ if $0 \leqslant y \leqslant 1$, as in [7].

For the second-order derivatives $\partial_x[\partial_x[u^\varepsilon * \rho_{h(\varepsilon)}]] * \rho^1_{h(\varepsilon)}$ and in order to retain symmetry, we consider $\rho^1(y) = 0$ if $y > 0$ and $\rho^1(y) = 1$ if $-1 \leqslant y \leqslant 0$. This allows us to obtain (with the notation $h(\varepsilon) = h$) Eq. (2) for a representative u^ε, which is simply a space-decentered discretization:

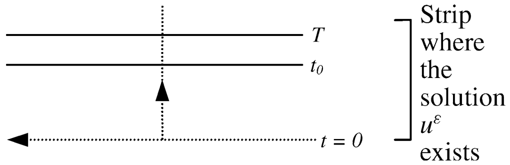
$$\begin{cases} \partial_t u^\varepsilon(x,t) = -a^\varepsilon(x)F\Big[\frac{u^\varepsilon(x,t)-2u^\varepsilon(x-h,t)+u^\varepsilon(x-2h,t)}{h^2}, \frac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\Big]\\ \qquad\qquad \times f'(u^\varepsilon(x,t))\Big(\frac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\Big),\\ u^\varepsilon(x,0) = u^{0,\varepsilon} \quad (\text{with } u^{0,\varepsilon} \text{ a } C^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}. \end{cases} \tag{3}$$

We will now show that for a fixed ε, Eq. (3) has a unique global solution in C^∞. To do so, we will first prove a maximum principle for u^ε. More precisely, we have the following theorem.

Theorem 4.1 (Maximum principle). *If u^ε, a solution of* (3), *exists on* $\mathbb{R}\times[0,T]$ *and if* $u^{0,\varepsilon}$ *vanishes when x is close to $-\infty$, then we have*

$$\inf u^{0,\varepsilon}(x) \leqslant u^\varepsilon(x,t) \leqslant \sup u^{0,\varepsilon}(x) \quad \text{on } \mathbb{R}\times[0,T].$$

Proof. Suppose that a $\mathcal{C}^1$ solution $u^\varepsilon(x,t)$ exists along the strip $\mathbb{R}\times[0,T]$; suppose that the initial condition $u^{0,\varepsilon}$ is continuous on a real axis $\mathbb{R}$ and tends to 0 in the neighborhood of $\pm\infty$. Let (x_0,t_0), with $0\leqslant t_0\leqslant T$, be such that $u^\varepsilon(x_0,t_0)=\sup\{u^\varepsilon(x,t), (x,t)\in\mathbb{R}\times[0,T]\}$.



Suppose now that $t_0\neq 0$. Since the maximum is reached on t_0 for a fixed $x=x_0$, by tending t toward t_0 with lower values we obtain the inequality

$$\partial_t u^\varepsilon(x_0,t_0)\geqslant 0 \quad (\text{even if } t_0=T). \tag{4}$$

However, the hypothesis $a^\varepsilon F^\varepsilon f'>0$ and Eq. (2) imply that

$$\partial_t u^\varepsilon(x_0,t_0)\leqslant 0. \tag{5}$$

Moreover, (4) and (5) imply that $\partial_t u^\varepsilon(x_0,t_0)=0$. The fact that $a^\varepsilon F f'$ is positive leads to

$$u^\varepsilon(x_0-\varepsilon,t_0)=u^\varepsilon(x_0,t_0). \tag{6}$$

Therefore, $(x_0-\varepsilon,t_0)$ also provides the maximum. We repeat the same argument and obtain that the maximum is reached on all points $(x_0-k\varepsilon,t_0)$, $k\in\mathbb{N}$ (the set of integer numbers), and therefore

$$u^\varepsilon(x_0-k\varepsilon,t_0)=u^\varepsilon(x_0,t_0). \tag{7}$$

We have supposed that $a^\varepsilon F^\varepsilon f'$ is uniformly bounded on $\mathbb{R}\times[0,T]$. We now suppose that $u^{0,\varepsilon}$ vanishes for $x<-\alpha$ for some $\alpha>0$ (this hypothesis can be replaced by: $u^{0,\varepsilon}$ tends to 0 when x tends to $-\infty$). Hence, there exists a real $\beta>0$ such that $u^\varepsilon(x,t)=0$ when $x>-\beta$ and $0\leqslant t\leqslant T$. (To see this, follow the characteristics; in this instance we can consider that $\beta=\alpha$ since the characteristics slopes are positive.) Thus, from (7) we obtain $u^\varepsilon(x_0,t_0)=0$, which leads to

$$u^\varepsilon(x,t)\leqslant 0, \quad \forall (x,t)\in\mathbb{R}\times[0,T]. \tag{8}$$

Particularly,

$$u^{0,\varepsilon}(x)\leqslant 0, \quad \forall x\in\mathbb{R}. \tag{9}$$

This contradicts the hypothesis $t_0\neq 0$ since $u^{0,\varepsilon}$ can take positive values. For inf, we proceed analogously and achieve

$$u^{0,\varepsilon}(x)\geqslant 0, \quad \forall x\in\mathbb{R}.$$

The hypothesis $t_0 \neq 0$ is then absurd because $u^{0,\varepsilon}$ can take negative values. Lastly, note that $u \equiv 0$ implies $u_0 \equiv 0$. The theorem has been proved in all cases. □

We will now give the global (with respect to time) existence and uniqueness theorem to problem (3).

Theorem 4.2 (Global existence and uniqueness solution to problem (3)). *For all* $\varepsilon > 0$, *and under the assumptions* $a \in L^\infty(\mathbb{R})$ *and* F *and* f *are* $\mathcal{C}^\infty$ *functions, Eq.* (3) *admits a global solution on* $[0, +\infty[$.

Proof. Let E be the Banach space of continuous functions from $\mathbb{R}$ into $\mathbb{R}$ that tend to 0 in the neighborhood of infinity (these functions are therefore bounded). This space is provided with the norm $\|f\| = \sup_{x\in\Re} |f(x)|$. We consider Eq. (3) as a differential equation in the space E according to the variable t. Indeed, set $X(t) = u^\varepsilon(\cdot, t) \in E$, $G(u, v) = a^\varepsilon F(u, v)$, and $\tau_h : E \to E$ is the translation operator defined as

$$\tau_h : E \to E, \quad \varphi(x) \to \varphi(x - h).$$

Then Eq. (3) can be written as

$$\begin{cases} X'(t) = -G\left(\frac{X(t)-2\tau_h X(t)+\tau_{2h}X}{h^2}, \frac{X(t)-\tau_h X(t)}{h}\right) f'(X(t))\left(\frac{X(t)-\tau_h X(t)}{h}\right), \\ X(0) = X_0 \quad (\text{where } X_0 = u^{0,\varepsilon}). \end{cases} \tag{10}$$

This equation is of the form

$$\begin{cases} X' = L(X(t), t), \\ X(t_0) = X_0, \end{cases}$$

with

$$L(X, t) = -G\left(\frac{X(t) - 2\tau_\varepsilon X(t) + \tau_{2\varepsilon} X}{h^2}, \frac{X(t) - \tau_\varepsilon X(t)}{h}\right) f'\big(X(t)\big) \times \left(\frac{X(t) - \tau_\varepsilon X(t)}{h}\right).$$

According to the classical theory of differential equations in Banach spaces (see Appendix A for an overview), if L is a Lipschitzian function with a Lipschitz constant bounded on every bounded set of E, the above equation admits a unique local solution $X(t)$. Furthermore, if $X(t)$ is globally bounded, it is a global solution. The following lemmas prove the required property on L. The global boundness of the local solution $X(t)$ comes from the maximum principle. This completes the proof of the theorem. □

Lemma 4.1. *The application* $(u, v) \to G(u, v)$ *is* $\mathcal{C}^\infty$ *from* $E \times E$ *into* E (*if* G *is a* $\mathcal{C}^\infty$ *function from* $\mathbb{R} \times \mathbb{R}$ *into* $\mathbb{R}$ *with* $G(0, 0) = 0$).

Lemma 4.2. *The application* $(X, Y) \to G(X, Y)$ *is Lipschitzian with a bounded Lipschitz constant on every bounded set of* E.

Lemma 4.3. *The application* $(u) \to f(u)$ *is* $\mathcal{C}^\infty$ *from* E *into* E (*if* f *is a* $\mathcal{C}^\infty$ *function from* $\mathbb{R}$ *into* $\mathbb{R}$ *with* $f(0) = 0$). *Moreover, it is a Lipschitzian function with a bounded Lipschitz constant on every bounded set of* E.

See Appendix B for the proofs.

Following the routine arguments and methods exposed in detail in [8] and widely used in the recent literature (examples are [7,10,14,17]), we prove that u^ε is in $\mathcal{E}_{M,\varepsilon}(\mathbb{R}^+ \times \mathbb{R})$. Its class U in $\mathcal{G}_{s,g}(\mathbb{R}^+ \times \mathbb{R})$ is then a solution of (2). The uniqueness can be established along the same lines as in [8].

Remark. The case of $a(\cdot)F(\cdot,\cdot)f'(\cdot) < 0$ is similar to previous one, but with the cone of dependency oriented to the right. As such, we choose $\rho(y) = 0$ if $y > 0$ and $\rho(y) = 1$ if $-1 \leqslant y \leqslant 0$ as a mollifier for the first-order derivatives, and retain the same choices for the second-order derivatives in order to obtain the following equation:

$$\begin{cases} \partial_t u^\varepsilon(x,t) = -a^\varepsilon(x) F\left(\frac{u^\varepsilon(x+h,t)-2u^\varepsilon(x,t)+u^\varepsilon(x-h,t)}{h^2}, \frac{u^\varepsilon(x+h,t)-u^\varepsilon(x,t)}{h}\right) \\ \qquad\qquad \times f'(u^\varepsilon(x,t))\left(\frac{u^\varepsilon(x+h,t)-u^\varepsilon(x,t)}{h}\right), \\ u^\varepsilon(x,0) = u^{0,\varepsilon} \quad (\text{with } u^{0,\varepsilon} \text{ a } \mathcal{C}^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}). \end{cases} \tag{11}$$

Let us now refer to the general case where the sign of $a(\cdot)F(\cdot,\cdot)f'(\cdot)$ is unknown. Reformulating the equation

$$U_t + A(x)F(U_{xx}, U_x)f'(U)\bar{\partial}_x U(x) = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+,$$

allows us to reduce it down to the case of positive slops. To do so, let us make a Galilean change of coordinates (uniform translation motion), and set

$$y = x - ct \quad \text{and} \quad \tau = t,$$

where c is a constant, whose value will be given below. Let

$$V(y,\tau) = U(x,t).$$

Then we obtain the following equation:

$$\partial_\tau V(y,\tau) + \left[A(y+c\tau)F\left(V_{yy}(y,\tau), V_y(y,\tau)\right)f'\left(V(y,\tau)\right) - c\right]\bar{\partial}_y V = 0,$$

and if v^ε is some representative of V, we have

$$\partial_\tau v^\varepsilon(y,\tau) + \left[a^\varepsilon(y+c\tau)F\left(v^\varepsilon_{yy}(y,\tau), v^\varepsilon_y(y,\tau)\right)f'\left(v^\varepsilon(y,\tau)\right) - c\right]\bar{\partial}_y v^\varepsilon = 0. \tag{12}$$

Now, assume that the initial condition u^0 is bounded on $\mathbb{R}$. In another words, there exists a constant M such that $-M < u^0(x) < M$ for all x in $\mathbb{R}$. This implies that $-M \leqslant v^\varepsilon \leqslant M$ (see proof below). We then choose the constant c such that

$$c < \inf\left\{a^\varepsilon(x)F(\lambda_1\lambda_2)f'(\lambda_3),\ -M < \lambda_1, \lambda_2, \lambda_3 < M,\ x \in \mathbb{R},\ \varepsilon > 0\right\}.$$

Thus, the quantity

$$\left[a^\varepsilon(y+c\tau)F\left(v^\varepsilon_{yy}(y,\tau), v^\varepsilon_y(y,\tau)\right)f'\left(v^\varepsilon(y,\tau)\right) - c\right]$$

is positive because $v^\varepsilon(y,\tau)$ belongs to the interval $[-M, M]$ for all y and τ.

We set

$$\Im(y,\tau,\lambda_1,\lambda_2,\lambda_3)=a^\varepsilon(y+c\tau)F(\lambda_1,\lambda_2)f(\lambda_3)-c\lambda_3.$$

Then we have

$$\partial_{\lambda_3}\Im>0,\quad \forall\lambda_1,\lambda_2,\lambda_3\in[-M,M].$$

This leads to the positive slope case studied previously. The following equation is then derived:

$$\begin{cases} \partial_t v^\varepsilon(y,\tau)=-\frac{1}{h}\Big[\partial_{\lambda_3}\Im\Big(y,\tau,\frac{v^\varepsilon(y,\tau)-2v^\varepsilon(y-h,\tau)+v^\varepsilon(y-2h,\tau)}{(h)^2}, \\ \qquad\qquad \frac{v^\varepsilon(y,\tau)-v^\varepsilon(y-h,\tau)}{h}v^\varepsilon(y,\tau)\Big)\Big] \\ \qquad\qquad \times(v^\varepsilon(y,\tau)-v^\varepsilon(y-h,\tau)), \\ v^\varepsilon(y,0)=v^{0,\varepsilon}\quad (\text{with } v^{0,\varepsilon} \text{ a } \mathcal{C}^\infty \text{ function and } v^0=\text{class of } v^{0,\varepsilon}). \end{cases} \tag{13}$$

Hence, the maximum principle (Theorem 4.1) and the existence and uniqueness of a global solution v^ε of (13) (Theorem 4.2) hold, and the class of v^ε in $\mathcal{G}_{s,g}(\mathbb{R}^+\times\mathbb{R})$ is a unique solution of (2). This is obviously true, while the assumption $-M\leqslant v^\varepsilon\leqslant M$ we have supposed up to now is also true. The following lemma prove this assumption.

Lemma 4.4. *If v^ε is a solution of Eq.* (12)*, then there exists a constant M such that $-M\leqslant v^\varepsilon\leqslant M$, $\forall\varepsilon>0$.*

Proof. The classical theory of differential equations provides the existence and uniqueness of a maximal solution v^ε of Eq. (12) on an interval $[0,T[$. Let T_1 be such that $T_1<T$. Restricted to the interval $[0,T_1]$, v^ε is bounded by a constant $M(\varepsilon,T_1)$ (that is dependent on ε and T_1). Note that this solution does not depend on the constant c (Galilean change of coordinates). Then we can choose c such that the characteristics slops of (12) are positive on $[0,T_1]$. Consequently, the arguments proving the maximum principle can be used to conclude that v^ε reaches its maximum and minimum on $[0,T_1]$ at 0. Therefore, $M(\varepsilon,T_1)$ does not depend on either ε or T_1. Furthermore, since T_1 is chosen arbitrarily, v^ε is bounded independently of ε on the entire interval $[0,T[$, and $T=+\infty$. □

In conclusion, we have established the following theorem.

Theorem 4.3. *Let u^0 and a belong to $L^\infty(\mathbb{R})$ and let f, F be $\mathcal{C}^\infty$ bounded functions. Let U^0 and A be the images in $\mathcal{G}_{\varepsilon,g}(\mathbb{R})$ of u^0 and a be the injection of $L^\infty(\Re)$ in $\mathcal{G}_{\varepsilon,g}(\mathbb{R})$ (obtained by a regularization process, as defined above). Assume that the scale function $h(\varepsilon)\to 0$ is sufficiently slow. Then Eq.* (2) *has a unique solution U in $\mathcal{G}_{s,g}(\mathbb{R}^+\times\mathbb{R})$. That is, U satisfies the following*:

$$\begin{cases} U_t+A(x)F(\bar\partial_{xx}U,\bar\partial_x U_x)\bar\partial_x f(U(x))=0, \\ U(0,\cdot)=U^0, \end{cases} \quad \textit{in } \mathbb{R}\times\mathbb{R}^+.$$

4.2. Viscous profile of Eq. (13)*, the model with initial variables*

The change of variable we made allows us to return to positive velocities and recover the theoretical results we have presented in this case. What is the outcome when Eq. (13) is re-expressed with the original variables? Equation (13) is given with the variables y and τ. If we return to the initial variables x and t, we obtain

$$v^\varepsilon(y,\tau) = u^\varepsilon(x,t) = u^\varepsilon(y + c\tau, \tau);$$

then

$$\partial_\tau v^\varepsilon(y,\tau) = c\partial_x u^\varepsilon(y + c\tau, \tau) + \partial_t u^\varepsilon(y + c\tau, \tau),$$
$$\partial_\tau v^\varepsilon(y,\tau) = c\partial_x u^\varepsilon(x,t) + \partial_t u^\varepsilon(x,t),$$

from which (13) becomes

$$\begin{aligned}\partial_t u^\varepsilon(x,t) = -a^\varepsilon(x) F\big(&\tfrac{u^\varepsilon(x,t)-2u^\varepsilon(x-h,t)+u^\varepsilon(x-2h,t)}{h^2}, \tfrac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\big) \\ &\times f'\big(u^\varepsilon(x,t)\big)\left(\frac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\right) \\ &+ c\left(\frac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h} - \partial_x u^\varepsilon(x,t)\right).\end{aligned}$$

The last term is in $\partial_{x^2} u^\varepsilon(x,t)$. The two-order Taylor expansion gives us

$$u^\varepsilon(x-h,t) = u^\varepsilon(x,t) - h\partial_x u^\varepsilon(x,t) + \frac{h^2}{2}\partial_{x^2} u^\varepsilon(x-\theta h, t) \quad \text{with } 0 < \theta < 1;$$

then

$$c\left(\frac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h} - \partial_x u^\varepsilon(x,t)\right) = -c\frac{h}{2}(\partial^2/\partial_{x^2})u^\varepsilon(x-\theta h, t).$$

Finally, we are concerned with the equation

$$\begin{cases} \partial_t u^\varepsilon(x,t) = -a^\varepsilon(x) F\big(\tfrac{u^\varepsilon(x,t)-2u^\varepsilon(x-h,t)+u^\varepsilon(x-2h,t)}{h^2}, \tfrac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\big) \\ \qquad\qquad \times f'(u^\varepsilon(x,t))\big(\tfrac{u^\varepsilon(x,t)-u^\varepsilon(x-h,t)}{h}\big) - c\tfrac{h}{2}\partial_{x^2} u^\varepsilon(x-\theta h,t), \\ \quad 0 < \theta < 1,\ \theta = \theta(x,t,h), \\ u^\varepsilon(x,0) = u^{0,\varepsilon} \text{ (with } u^{0,\varepsilon} \text{ a } \mathcal{C}^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}). \end{cases} \tag{14}$$

The new term is of a viscosity type because the constant c is negative. Hence, as mentioned in the introduction, the smoothing aspect (represented by the viscosity term) appears to be inherent to the model when it is interpreted in the generalized functions framework.

5. Two-dimensional proposed model (images)

We generalize the one-dimensional shock filter model to two-dimensional case (images) by considering the following hyperbolic quasi-linear equation:

$$\begin{aligned} &u_t + a_1 F_1(\Delta u, u_x)\partial_x f_1(u) + a_2 F_2(\Delta u, u_y)\partial_y f_2(u) = 0, \\ &u(0,x) = u^0. \end{aligned} \tag{15}$$

The coefficients a_1 and a_2 (which can be discontinuous) and functions F_1, F_2, f_1, and f_2 ($\mathcal{C}^\infty$ functions) play the same role as in the one-dimensional case. Globally, they control the propagation speed of the created shocks, and are able to provide selective treatment according to the damaged region.

After improving on the Osher and Rudin model, Alvarez and Mazorra [2] generalize their model to two-dimensions by developing the directional smoothing idea (all the gradient directions). In the present work, we generalize the proposed one-dimensional model without favoring any direction in particular. This decision is motivated by several factors. The first is a theoretical motivation, which consists of the simple generalization of the theoretical study of the one-dimensional case, and benefiting from all that is offered by the generalized functions theory. We can mention, for instance, the very important existence and uniqueness result that renders the problem well-posed, the construction of simple and efficient numerical schemes and a precise adaptation of the model (when interpreted in the generalized functions framework) for the enhancement and restoration problem. The second motivation is that, in practice, the results obtained using the directional and non-directional derivatives are similar overall. In the following sections, we will develop the results established for the one-dimensional case without going into great detail since the proofs are comparable those from the previous case.

5.1. Existence and uniqueness of a generalized solution

We rewrite Eq. (15) as in the one-dimensional case using the regularized derivatives concept and replacing the initial condition u^0 and the coefficients a_1 and a_2 by an associate generalized functions obtained by the regularized process described above. We thus obtain the following equation:

$$U_1 + A_1 F_1(\bar{\Delta} U, \bar{\partial}_x U) f_1'(U) \bar{\partial}_x U + A_2 F_2(\bar{\Delta} U, \bar{\partial}_y U) f_2'(U) \bar{\partial}_y U = 0 \quad \text{in } \mathcal{G}$$
$$\text{and} \quad \big((x,y),t\big) \in \mathbb{R}^2 \times \mathbb{R}^+, \tag{16}$$

where

$$\bar{\Delta} = \bar{\partial}_x + \bar{\partial}_y, \qquad A_1 = \text{class}\{a_1^\varepsilon\}, \qquad A_2 = \text{class}\{a_2^\varepsilon\}, \qquad U^0 = \text{class}\{u_0^\varepsilon\}.$$

As in the one-dimensional case, and if we refer by u^ε to some representative of the generalized function U, u^ε satisfy one of the following equations according to the sign of the characteristics slops $a_1(\cdot)F_1(\cdot,\cdot)f_1'(\cdot)$ and $a_2(\cdot)F_2(\cdot,\cdot)f_2'(\cdot)$; that is, if $a_1(\cdot)F_1(\cdot,\cdot) \times f_1'(\cdot) > 0$ and $a_2(\cdot)F_2(\cdot,\cdot)f_2'(\cdot) > 0$, u^ε satisfy for a fixed ε the following equation:

$$\partial_t u^\varepsilon(x,t) = -a_1^\varepsilon(x,y) F_1 \left(\begin{array}{c} \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x-h_1,y,t) + u^\varepsilon(x-2h_1,y,t)}{h_1^2} \\ + \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x,y-h_2,t) + u^\varepsilon(x_1,y-2h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x,y,t) - u^\varepsilon(x-h_1,y,t)}{h_1} \end{array} \right)$$
$$\times f_1'\big(u^\varepsilon(x,y,t)\big)\left(\frac{u^\varepsilon(x,y,t) - u^\varepsilon(x-h_1,y,t)}{h_1}\right)$$

$$- a_2^\varepsilon(x,y) F_2 \begin{pmatrix} \frac{u^\varepsilon(x,y,t)-2u^\varepsilon(x-h_1,y,t)+u^\varepsilon(x-2h_1,y,t)}{h_1^2} \\ \quad + \frac{u^\varepsilon(x,y,t)-2u^\varepsilon(x,y-h_2,t)+u^\varepsilon(x_1,y-2h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x,y,t)-u^\varepsilon(x,y-h_2,t)}{h_2} \end{pmatrix}$$

$$\times f_2'\bigl(u^\varepsilon(x,y,t)\bigr)\left(\frac{u^\varepsilon(x,y,t)-u^\varepsilon(x,y-h_2,t)}{h_2}\right),$$

$$u^\varepsilon(x,y,0) = u^{0,\varepsilon} \quad (\text{with } u^{0,\varepsilon} \text{ a } C^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}). \tag{17}$$

We can simplify further by making $h_1 = h_2$. If $a_1(\cdot)F_1(\cdot,\cdot)f_1'(\cdot) < 0$ and $a_2(\cdot)F_2(\cdot,\cdot) \times f_2'(\cdot) < 0$, this case is similar to the previous one. As in the one-dimensional case, we obtain the following equation:

$$\partial_t u^\varepsilon(x,t) = -a_1^\varepsilon(x,y) F_1 \begin{pmatrix} \frac{u^\varepsilon(x+h_1,y,t)-2u^\varepsilon(x,y,t)+u^\varepsilon(x-h_1,y,t)}{h_1^2} \\ \quad + \frac{u^\varepsilon(x,y+h_2,t)-2u^\varepsilon(x,y,t)+u^\varepsilon(x_1,y-h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x+h_1,y,t)-u^\varepsilon(x,y,t)}{h_1} \end{pmatrix}$$

$$\times f_1'\bigl(u^\varepsilon(x,y,t)\bigr)\left(\frac{u^\varepsilon(x+h_1,y,t)-u^\varepsilon(x,y,t)}{h_1}\right)$$

$$- a_2^\varepsilon(x,y) F_2 \begin{pmatrix} \frac{u^\varepsilon(x+h_1,y,t)-2u^\varepsilon(x,y,t)+u^\varepsilon(x-h_1,y,t)}{h_1^2} \\ \quad + \frac{u^\varepsilon(x,y+h_2,t)-2u^\varepsilon(x,y,t)+u^\varepsilon(x_1,y-h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x,y+h_2,t)-u^\varepsilon(x,y,t)}{h_2} \end{pmatrix}$$

$$\times f_2'\bigl(u^\varepsilon(x,y,t)\bigr)\left(\frac{u^\varepsilon(x,y+h_2,t)-u^\varepsilon(x,y,t)}{h_2}\right),$$

$$u^\varepsilon(x,y,0) = u^{0,\varepsilon} \quad (\text{with } u^{0,\varepsilon} \text{ a } \mathcal{C}^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}). \tag{18}$$

If $a_1(\cdot)F_1(\cdot,\cdot)f_1'(\cdot)$ and $a_2(\cdot)F_2(\cdot,\cdot)f_2'(\cdot)$ have an unknown sign, we proceed as in the one-dimensional case by reformulating Eq. (16) using the Galilean transform $x = r - c_1 t$, $y = s - c_2 t$, and $\tau = t$ with

$$c_1 < \inf\bigl\{a_1^\varepsilon(x,y)F_1(\lambda_1,\lambda_2)f_1'(\lambda_3),\ -M < \lambda_1,\lambda_2,\lambda_3 < M,\ x,y \in \mathbb{R},\ \varepsilon > 0\bigr\}$$

and

$$c_2 < \inf\bigl\{a_2^\varepsilon(x,y)F_2(\lambda_1,\lambda_2)f_2'(\lambda_3),\ -M < \lambda_1,\lambda_2,\lambda_3 < M,\ x,y \in \mathbb{R},\ \varepsilon > 0\bigr\},$$

where M is a constant such that $-M < u^0(x,y) < M$ for every (x,y) in $\mathbb{R}^2$. Set $V(r,s,\tau) = U(x,y,t)$ and similarly to the one-dimensional case, we have for some representative v^ε of V the following equation:

$$\begin{aligned} &\partial_\tau v^\varepsilon(r,s,\tau) \\ &\quad + \bigl[a_1^\varepsilon(r+c_1\tau, s+c_2\tau)F_1\bigl(\Delta v^\varepsilon(r,s,\tau), v_r^\varepsilon(r,s,\tau)\bigr)f_1'(v^\varepsilon) - c_1\bigr]\bar{\partial}_r v^\varepsilon \\ &\quad + \bigl[a_2^\varepsilon(r+c_1\tau, s+c_2\tau)F_2\bigl(\Delta v^\varepsilon(r,s,\tau), v_s^\varepsilon(r,s,\tau)\bigr)f_2'(v) - c_2\bigr]\bar{\partial}_s v^\varepsilon = 0. \end{aligned}$$

Set

$$\Im_1(r,s,\tau,\lambda_1,\lambda_2,\lambda_3) = a_1^\varepsilon(r+c_1\tau, s+c_2\tau)F_1(\lambda_1,\lambda_2)f_1(\lambda_3) - c\lambda_3,$$
$$\Im_2(r,s,\tau,\lambda_1,\lambda_2,\lambda_3) = a_2^\varepsilon(r+c_1\tau, s+c_2\tau)F_2(\lambda_1,\lambda_2)f_2(\lambda_3) - c\lambda_3.$$

With the above choices for c_1 and c_2, the quantities

$$\left[a_1^\varepsilon(r+c_1\tau, s+c_2\tau)F_1\bigl(\Delta v^\varepsilon(r,s,\tau), v_r^\varepsilon(r,s,\tau)\bigr)f_1'(v^\varepsilon) - c_1\right]$$

and

$$\left[a_2^\varepsilon(r+c_1\tau, s+c_2\tau)F_2\bigl(\Delta v^\varepsilon(r,s,\tau), v_s^\varepsilon(r,s,\tau)\bigr)f_2'(v) - c_2\right]$$

are positive since $v(r,s,\tau)$ belongs in $[-M,M]$ for all r,s, and τ (the proof of this assertion is similar to the one-dimensional case). Thus, we have $\partial_{\lambda_3}\Im_1 > 0$ and $\partial_{\lambda_3}\Im_2 > 0$, $\forall\lambda_1,\lambda_2,\lambda_3 \in [-M,M]$, which leads to the positive slopes case and then the following equation is derived:

$$\begin{aligned}
\partial_\tau v^\varepsilon(r,s,\tau) = {}& -\frac{1}{h_1}\left[\partial_{\lambda_3}\Im_1\begin{pmatrix} r,s,\tau,\frac{v^\varepsilon(r,s,\tau)-2v^\varepsilon(r-h_1,s,\tau)+v^\varepsilon(r-2h_1,s,\tau)}{h_1^2} \\ +\frac{v^\varepsilon(r,s,\tau)-2v^\varepsilon(r,s-h_2,\tau)+v^\varepsilon(r,s-2h_2,\tau)}{h_2^2}, \\ \frac{v^\varepsilon(r,s,\tau)-v^\varepsilon(r-h_1,s,\tau)}{h_1}, v^\varepsilon(r,s,\tau)\end{pmatrix}\right. \\
&\left.\times\bigl(v^\varepsilon(r,s,\tau)-v^\varepsilon(r-h_1,s,\tau)\bigr)\right] \\
&-\frac{1}{h_2}\left[\partial_{\lambda_3}\Im_2\begin{pmatrix} r,s,\tau,\frac{v^\varepsilon(r,s,\tau)-2v^\varepsilon(r-h_1,s,\tau)+v^\varepsilon(r-2h_1,s,\tau)}{h_1^2} \\ +\frac{v^\varepsilon(r,s,\tau)-2v^\varepsilon(r,s-h_2,\tau)+v^\varepsilon(r,s-2h_2,\tau)}{h_2^2}, \\ \frac{v^\varepsilon(r,s,\tau)-v^\varepsilon(r,s-h_2,\tau)}{h_2}, v^\varepsilon(r,s,\tau)\end{pmatrix}\right. \\
&\left.\times\bigl(v^\varepsilon(r,s,\tau)-v^\varepsilon(r,s-h_2,\tau)\bigr)\right],
\end{aligned}$$

$$v^\varepsilon(r,s,0) = u^{0,\varepsilon} \quad \text{(with } u^{0,\varepsilon} \text{ a } \mathcal{C}^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}\text{).} \tag{19}$$

Equation (19) is a semi-discretization in space of Eq. (15).

The maximum principle and global existence and uniqueness theorems of the one-dimensional case are generalized to the two-dimensional case.

Theorem 5.1 (Maximum principle). *If u^ε, a solution of* (17), (18), *or* (19) *exists on* $\mathbb{R}^2 \times [0,T]$ *and if $u^{0,\varepsilon}$ vanishes when x or y approaches $-\infty$, we have*

$$\inf u^{0,\varepsilon}(x,y) \leqslant u^\varepsilon(x,y,t) \leqslant \sup u^{0,\varepsilon}(x,y) \quad \textit{on } \mathbb{R}^2 \times [0,T].$$

Theorem 5.2 (Global existence and uniqueness solution to problem (16)). *For all $\varepsilon > 0$, Eqs.* (17)–(19) *admit a global solution on* $[0,+\infty[$.

The proofs of these two theorems are similar to the one-dimensional case. Thus the details of these proofs are not provided in this paper.

With the same arguments as in the one-dimensional case, the class of v^ε (of u^ε in the case of unchanged slopes sign) in $\mathcal{G}_{s,g}(\mathbb{R}^+ \times \mathbb{R}^2)$ is a unique solution of (16).

In conclusion, we have established the following theorem.

Theorem 5.3. *Let u^0, a_1, and a_2 belong in $L^\infty(\Re^2)$ and let f_1, f_2, F_1, and F_2 be $\mathcal{C}^\infty$ bounded functions. Let U^0, A_1, and A_2 be the images in $\mathcal{G}_{s,g}(\mathbb{R}^2)$ of u^0, a_1, and a_2 be the injection of $L^\infty(\mathbb{R})$ in $\mathcal{G}_{s,g}(\mathbb{R}^2)$ (obtained by a regularization process, as defined above). Assume that the scale function $h(\varepsilon) \to 0$ is sufficiently slow. Then Eq.* (16) *has a unique equation U in $\mathcal{G}_{s,g}(\mathbb{R}^+ \times \mathbb{R}^2)$. That is, U satisfies the following*:

$$\begin{cases} U_t + A_1 F_1(\bar{\Delta} U, \bar{\partial}_x U)\bar{\partial}_x f_1(U) + A_2 F_2(\bar{\Delta} U, \bar{\partial}_y U)\bar{\partial}_y f_2(U) = 0 \quad \text{in } \mathcal{G} \\ \text{and} \quad ((x,y),t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ U^0(0,\cdot) = U^0. \end{cases}$$

5.2. Viscous profile

As in the one-dimensional case, the Galilean transform allows us to return to the characteristic slopes. We shall see that the two-dimensional case also involves contending with a viscous profile. Equation (19) is expressed with the variables r, s, and τ; let us instead express it with variables x, y, and t. We have

$$v^\varepsilon(r,s,\tau) = u^\varepsilon(x,y,t) = u^\varepsilon(r + c_1\tau, s + c_2\tau, \tau),$$

and then

$$\partial_\tau v^\varepsilon(r,s,\tau) = c_1 \partial_x u^\varepsilon(x,y,t) + c_2 \partial_y u^\varepsilon(x,y,t) + \partial_t u^\varepsilon(x,y,t).$$

By replacing in (19) and using the Taylor formula up to order two as in the one-dimensional case, the equation once reworked becomes

$$\begin{aligned}
\partial_t u^\varepsilon(x,t) = {}& -a_1^\varepsilon(x,y) F_1 \left(\begin{array}{l} \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x-h_1,y,t) + u^\varepsilon(x-2h_1,y,t)}{h_1^2} \\ \quad + \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x,y-h_2,t) + u^\varepsilon(x_1,y-2h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x,y,t) - u^\varepsilon(x-h_1,y,t)}{h_1} \end{array} \right) \\
& \times f_1'\big(u^\varepsilon(x,y,t)\big)\left(\frac{u^\varepsilon(x,y,t) - u^\varepsilon(x-h_1,y,t)}{h_1}\right) \\
& - c_1 \frac{h_1}{2} \partial_{x^2} u^\varepsilon(x - \theta_1 h_1, y, t) \quad \big(0 < \theta_1 < 1,\ \theta_1 = \theta_1(x,y,t,h_1)\big) \\
& - a_2^\varepsilon(x,y) F_2 \left(\begin{array}{l} \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x-h_1,y,t) + u^\varepsilon(x-2h_1,y,t)}{h_1^2} \\ \quad + \frac{u^\varepsilon(x,y,t) - 2u^\varepsilon(x,y-h_2,t) + u^\varepsilon(x_1,y-2h_2,t)}{h_2^2}, \\ \frac{u^\varepsilon(x,y,t) - u^\varepsilon(x,y-h_2,t)}{h_2} \end{array} \right) \\
& \times f_2'\big(u^\varepsilon(x,y,t)\big)\left(\frac{u^\varepsilon(x,y,t) - u^\varepsilon(x,y-h_2,t)}{h_2}\right)
\end{aligned}$$

$$- c_2 \frac{h_2}{2} \partial_{y^2} u^\varepsilon(x, y - \theta_2 h_2, t) \quad \big(0 < \theta_2 < 1,\ \theta_2 = \theta_2(x, y, t, h_2)\big),$$

$$u^\varepsilon(x, y, 0) = u^{0,\varepsilon} \quad (\text{with } u^{0,\varepsilon} \text{ a } C^\infty \text{ function and } u^0 = \text{class of } u^{0,\varepsilon}).$$

Since constants c_1 and c_2 are negative, the appeared term is of the viscous type.

6. Conclusion

In Part I of this paper, we proposed generalized one- and two-dimensional shock models for signal enhancement and restoration where the shock propagation speed is well controlled. This generalization makes the model more robust and efficient for a large scope of applications. After an interpretation of the proposed models in a recently-developed framework of generalized functions algebra and using the regularized derivatives concept, we proved an existence and uniqueness of solutions result, which makes the models well-posed. Beyond the theoretical tools offered by the generalized functions framework, there is an appreciable practical impact on signal restoration. First recall that the signal restoration is achieved by taking the signal as the initial condition of the model and the steady state of the model is taken as the restored signal. In general, the input signals are noisy, and obtaining restored and free noise signals is important in the field of signal process. Since we deal with representatives (the initial condition is regularized) when we interpret the model within the framework of generalized functions algebra, almost the noise is removed at the beginning of the process. More than that, the viscous profile of the models helps to remove the rest of the noise during the process. The proved maximum principle, in addition to its usefulness in the existence and uniqueness of a solution proof, shows that the output of a processed 256 grey-level image is still a 256 grey-level image without normalization each time, which in practice can cause some image distortion (recall that a grey-level image is a two-dimensional function which values are between 0 and 255). In Part II of this paper, we will investigate the numerical analysis aspect of the models and show some tests on one-dimensional signals and images.

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Appendix A. Overview of the classical theory of differential equations in Banach spaces

Let

$$\begin{cases} X' = L(X(t), t), \\ X(t_0) = X_0 \end{cases}$$

be a differential equation defined on E, where L is a Lipschitzian function with a Lipschitz constant K (i.e., $\|L(X,t) - L(y,t)\| \leqslant K\|X - Y\|$). Suppose that K is bounded on every bounded set of E (not only locally). Thus, if at the same time L is defined on $E \times \Re$, the maximal solution of the differential equation is defined on an interval $]T_1, T_2[$ such that $\|X(t)\|_E \to +\infty$ if $t \to T_1$, and likewise when $t \to T_2$. In other words, the solution "explodes," i.e., it ceases to exist only when $X(t)$ tends to infinity. In the case of a finite-dimension space E (here, K is necessarily bounded on every bounded set of E if L is $\mathcal{C}^1$ because bounded implies relatively compact in finite dimensions), this result is classical. However, only the boundedness hypothesis cited above on K is truly used (and not the finite dimension of E). Consequently, if K is bounded on every bounded set of E and if we know that

$$\exists M > 0 \text{ such that } \|X(t)\|_E \leqslant M \text{ if } X(t) \text{ is a solution of } \begin{cases} X' = L(X(t), t), \\ X(t_0) = X_0, \end{cases}$$

defined on a given interval, then the equation admits a global solution.

Appendix B. Proofs of Lemmas 4.1, 4.2, and 4.3

In this part we recall the lemmas and give the proofs.

Lemma 4.1. *The application $(u, v) \to G(u, v)$ is $\mathcal{C}^\infty$ from $E \times E$ into E (if G is a $\mathcal{C}^\infty$ function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ with $G(0, 0) = 0$).*

Proof. Let $u, \xi_1, v, \xi_2 \in E$; then $G(u + \xi_1, v + \xi_2)$ is the real function such that

$$\mathbb{R} \to \mathbb{R}, \quad x \to G\big(X(x) + \xi_1(x), Y(x) + \xi_2(x)\big).$$

The second-order Taylor expansion gives

$$\begin{aligned} &G\big(u(x) + \xi_1(x), v(x) + \xi_2(x)\big) \\ &\quad = G\big(u(x), v(x)\big) + \partial_x G\big(u(x), v(x)\big) \cdot \xi_1(x) + \partial_y G\big(u(x), v(x)\big) \cdot \xi_2(x) \\ &\qquad + \frac{1}{2}\partial_x\partial_y G\big(u(x) + \theta_1(x)\xi_1(x), v(x) + \theta_2(x)\xi_2(x)\big) \cdot \xi_1(x)\xi_2(x) \\ &\qquad + \frac{1}{2}\partial_{xx} G\big(u(x) + \theta_1(x)\xi_1(x), v(x) + \theta_2(x)\xi_2(x)\big) \cdot \xi_1^2(x) \\ &\qquad + \frac{1}{2}\partial_{yy} G\big(u(x) + \theta_1(x)\xi_1(x), v(x) + \theta_2(x)\xi_2(x)\big) \cdot \xi_2^2(x), \\ &\quad 0 < \theta_1(x) < 1,\ 0 < \theta_2(x) < 1. \end{aligned}$$

For a fixed (X, Y), the application $(\xi_1, \xi_2) \to \partial_x G(X, Y) \cdot \xi_1 + \partial_y G(X, Y) \cdot \xi_1$ is linear continuous and the quantities $\partial_{xx} G(X + \xi_1, Y + \xi_2)$, $\partial_{xy} G(X + \xi_1, Y + \xi_2)$, and $\partial_{yy} G(X + \xi_1, Y + \xi_2)$ are bounded for $\|\xi_1\| < 1$ and $\|\xi_2\| < 1$. This demonstrates the lemma. □

Lemma 4.2. *The application $(X, Y) \to G(X, Y)$ is Lipschitzian with a bounded Lipschitz constant on every bounded set of E.*

Proof. The proof builds on the proof of previews lemma. If $\|X\| + \|Y\| < A$ for a given positive value A, we have

$$\left|\partial_x G\big(X(x), Y(x)\big)\right| < \sup_{\|X_1\|+\|Y_1\|<A} \partial_x G(X_1, Y_1) < K_1$$

and

$$\left|\partial_y G\big(X(x), Y(x)\big)\right| < \sup_{\|X_1\|+\|Y_1\|<A} \partial_y G(X_1, Y_1) < K_2,$$

where K_1 and K_2 are two constants, which proves the lemma. □

Lemma 4.3. *The application $(u) \to f(u)$ is $\mathcal{C}^\infty$ from E into E (if f is a $\mathcal{C}^\infty$ function from $\mathbb{R}$ into $\mathbb{R}$ with $f(0) = 0$). Moreover, it is a Lipschitzian function with a bounded Lipschitz constant on every bounded set of E.*

The proof of this lemma is analogous to Lemmas 4.1 and 4.2.

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