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Working and Assembly Modes of the Agile Eye

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***Abstract*— This paper deals with the in-depth kinematic analysis of a special parallel wrist, called the *Agile Eye*. The *Agile Eye* is a three-legged spherical parallel robot with revolute joints, in which all pairs of adjacent joint axes are orthogonal. Its most peculiar feature, demonstrated in this paper for the first time, is that its workspace is unlimited and flawed only by six singularity curves (instead of surfaces). These curves correspond to self-motions of the mobile platform and of the legs, or to a lockup configuration. This paper also demonstrates that the four solutions to the direct kinematics of the *Agile Eye* (assembly modes) have a simple direct relationship with the eight solutions to the inverse kinematics (working modes).**

I. Introduction

Most of the research work carried out in the field of parallel robots has been focused on a particularly challenging problem, namely, solving the direct kinematics, that it to say, finding each possible position and orientation (*pose*) of the mobile platform (*assembly mode*) as a function of the active-joint variables. A second popular problem has been the workspace evaluation and optimization [1], [2]. Unfortunately, the direct kinematic problem and the workspace analysis have most often been treated independently, although they are closely interrelated.

It is well known that most parallel robots have singularities in their workspace, where stiffness is lost [3]. These singularities (called *parallel singularities* in [4] and Type 2 in [3]) correspond to the set of active-joint variables for which the finite number of different direct kinematic solutions (assembly modes) changes. For parallel manipulators with several solutions to the inverse kinematic problem (*working modes*), another type of singularity exist and defines what may be generalized as the workspace boundary. These singularities (called *serial singularities* in [4] and Type 1 in [3]) correspond to the set of poses for which the finite number of different inverse kinematic solutions (working modes) changes.

While ensuring that a parallel robot stays with the same working mode is straightforward, the notion of "same assembly mode" is not even clear in the general case. Indeed, the direct kinematic solutions are most often obtained by solving a univariate polynomial of degree $n > 3$, which means that there is no way to designate each solution to a particular assembly mode. Then, how does one choose a direct kinematic solution for a given set of active-joint variables?

A natural sorting criterion is that the direct kinematic solution should be reachable through continuous motion from the initial assembly mode, i.e., the reference configuration of the robot, without crossing a singularity. Prior to [5], it was commonly believed that such a criterion was sufficient to determine a unique solution. Unfortunately, it was discovered that a non-singular configuration changing trajectory may exist between two assembly modes for so-called *cuspidal robots* [5], [6] . This result gave rise to a theoretical work on the concepts of characteristic surfaces and uniqueness domains in the workspace [7]. However, it still remains unknown what design parameters make a given parallel robot cuspidal, and it is still unclear how to make a given parallel robot work in the same assembly mode. It will be shown in this paper, that there is a spherical parallel robot (possibly the only one), for which there are clear answers to these complex questions.

In the family of parallel wrists [8], the *Agile Eye* provides high stiffness and is quite probably the only one to provide a theoretically *unlimited and undivided orientation workspace*. The *Agile Eye* is a 3-*R̲RR* spherical parallel mechanism in which the axes of all pairs of adjacent joints are orthogonal. Based on the *Agile Eye* design, a camera-orienting device was constructed at Laval University a decade ago [9], an orientable machine worktable was manufactured at Tianjin University, and a wrist for a 6-DOF robot was built at McGill University.

The *Agile Eye* has been extensively studied, but surprisingly some of its most interesting features have been ignored. One of the key references in this paper is [10], where the simple solution to the direct kinematics of the *Agile Eye* is presented. Namely, it is shown that the *Agile Eye* has always four trivial solutions (at which all three legs can freely rotate) and four nontrivial solutions obtained in cascade. A second key reference is [11], where the singularities of the *Agile Eye* are analyzed. Unfortunately, in [11], it was mistakenly assumed that the only singular configurations are the four trivial solutions to the direct kinematics. The fact that at each of these four orientations, a special arrangement of the legs allows the mobile platform to freely rotate about an axis was overlooked. Indeed, as this paper shows, the singularities of the *Agile Eye* are six curves in the three-dimensional workspace, corresponding to self-motions and lockup configurations.

When it was realized that the unlimited workspace of the *Agile Eye* is not divided by singularity surfaces (as is the case for all other 3-DOF parallel wrists), yet is shared by four unique assembly modes, it first seemed that the robot is cuspidal. However, this paper shows the contrary and proves

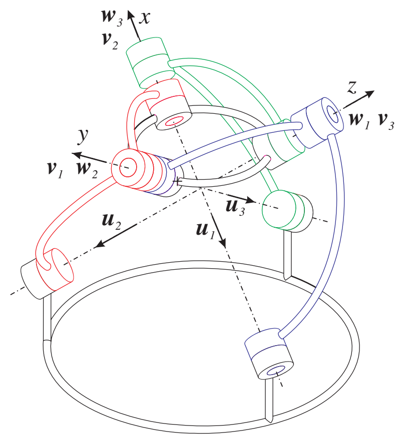


Fig. 1. The *Agile Eye* at its reference configuration.

that each assembly mode simply corresponds to a pair of working modes. The results of this paper provide further insight into the kinematics of the *Agile Eye* and shed light to the open problem of assembly mode designation.

The rest of the paper is organized as follows. In the next two sections, the kinematic model of the *Agile Eye* and its inverse kinematics are briefly presented. Then, the direct kinematics and the singularity analysis are addressed by partially reproducing, reformulating, and augmenting the results of [10] and [11]. Finally, it is shown that for each working mode, there is a single corresponding assembly mode.

II. Kinematic Model of the Agile Eye

Figure 1 depicts the *Agile Eye* at its reference configuration, where the mobile platform is at its zero orientation and all active-joint variables are zero. At this configuration, the axes of the first base joint, the second intermediate joint, and the third platform joint coincide, and so on for the other joints.

A base reference frame $Oxyz$ is chosen in such a way that its x axis is along the axis of the first base joint, its y axis is along the axis of the second base joint, and its z axis is along the axis of the third base joint. A mobile reference frame (not shown) is fixed at the mobile platform so that it coincides with the base frame at the zero orientation. Thus, the axes of the base joints are defined in the base frame as:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

Similarly, the axes of the platform joints are defined in the mobile frame as:

$$\mathbf{v}_1' = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2' = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}_3' = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

The rotation matrix $\mathbf{R}$ describes the orientation of the mobile frame with respect to the base frame. The *ZYX* Euler-angle convention is used here because it leads to simple results. In the base frame, the axes of the platform joints are defined as

$$\mathbf{v}_i = \mathbf{R}\mathbf{v}_i' = \mathbf{R}_z(\phi)\mathbf{R}_y(\theta)\mathbf{R}_x(\psi)\mathbf{v}_i', \tag{3}$$

where $i = 1, 2, 3$, which yields

$$\mathbf{v}_1 = \begin{bmatrix} \sin\phi\cos\psi - \cos\phi\sin\theta\sin\psi \\ -\cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi \\ -\cos\theta\sin\psi \end{bmatrix},$$
$$\mathbf{v}_2 = \begin{bmatrix} -\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi \\ \cos\phi\sin\psi - \sin\phi\sin\theta\cos\psi \\ -\cos\theta\cos\psi \end{bmatrix},$$
$$\mathbf{v}_3 = \begin{bmatrix} -\cos\phi\cos\theta \\ -\sin\phi\cos\theta \\ \sin\theta \end{bmatrix}. \tag{4}$$

Finally, the axes of the intermediate joints are defined in the base frame as:

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}, \ \mathbf{w}_2 = \begin{bmatrix} \cos\theta_2 \\ 0 \\ -\sin\theta_2 \end{bmatrix}, \ \mathbf{w}_3 = \begin{bmatrix} -\sin\theta_3 \\ \cos\theta_3 \\ 0 \end{bmatrix}, \tag{5}$$

where θ_i is the *active-joint variable* for leg i ($i = 1, 2, 3$).

III. Inverse Kinematics and Working Modes

Although very simple, the solutions to the inverse kinematics of the *Agile Eye* will be presented for completeness [10]. For a given orientation of the mobile platform, each leg admits two solutions for θ_i in $(-\pi, \pi]$ obtained from

$$\tan\theta_1 = \frac{\cos\theta\sin\psi}{\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi}, \tag{6}$$
$$\tan\theta_2 = \frac{\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi}{\cos\theta\cos\psi}, \tag{7}$$
$$\tan\theta_3 = \tan\phi. \tag{8}$$

Thus, the inverse kinematics admit eight real solutions (working modes) for any platform orientation. It is important to note that in the two solutions for θ_i, vector $\mathbf{w}_i$ only changes direction. When leg i is fully extended or folded, the corresponding equation from the above three ones does not hold true and θ_i can be arbitrary.

IV. Direct Kinematics and Assembly Modes

The kinematic design of the *Agile Eye* was arrived at by optimizing 3-*R̲RR* parallel wrists in search of maximum workspace and global dexterity index [9]. Incidentally, such properties also yield great simplification in the direct kinematics since four of the eight solutions become trivial. The direct kinematic problem of the *Agile Eye* was solved in [10] and will be reformulated and further analyzed here.

The following constraint equations are written:

$$\mathbf{w}_i^T\mathbf{v}_i = 0, \tag{9a}$$

thus,

$$\sin\psi(\sin\theta_1\sin\theta\sin\phi - \cos\theta\cos\theta_1) +$$
$$\cos\psi\sin\theta_1\cos\phi = 0, \tag{9b}$$
$$\cos\psi(\cos\theta_2\sin\theta\cos\phi - \cos\theta\sin\theta_2) +$$
$$\sin\psi\cos\theta_2\sin\phi = 0, \tag{9c}$$
$$\sin(\theta_3 - \phi)\cos\theta = 0. \tag{9d}$$

From (9d), the direct kinematic problem is found to admit two sets of solutions, defined by

$$\cos\theta = 0, \text{ and} \tag{10a}$$

$$\sin(\theta_3 - \phi) = 0. \tag{10b}$$

In the next two subsections, these two equations will be solved.

A. First Set of Solutions — Trivial Solutions

Equation (10a) gives two solutions for the angle θ,

$$\theta = \pi/2 \quad \text{and} \quad \theta = -\pi/2, \tag{11}$$

which correspond to XYZ Euler angle representation singularities. Using the first solution in (9b) and simplifying yields the following condition for arbitrary active-joint variables,

$$\cos(\phi - \psi) = 0, \tag{12}$$

and from the second solution,

$$\cos(\phi + \psi) = 0. \tag{13}$$

In both cases, each of these two equations lead to two solutions. Thus, because of the representation singularity, only four rotation matrices describe the corresponding orientations:

$$\mathbf{R}_{TO1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \ \mathbf{R}_{TO2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$
$$\mathbf{R}_{TO3} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \ \mathbf{R}_{TO4} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Figure 2 depicts the four trivial solutions (orientations) to the direct kinematic problem. It may be seen geometrically, as well as from (9b-d), that when the platform is at one of these four orientations, all three legs are singular (fully extended or folded) and can freely rotate about their base joint axes. Thus, these four orientations are trivial solutions to the direct kinematic problem of the *Agile Eye* and exist for any set of active-joint variables. These four families of configurations will be discussed further in the section on singularity analysis.

B. Second Set of Solutions — Nontrivial Solutions

Equation (10b) gives two solutions for the angle ϕ:

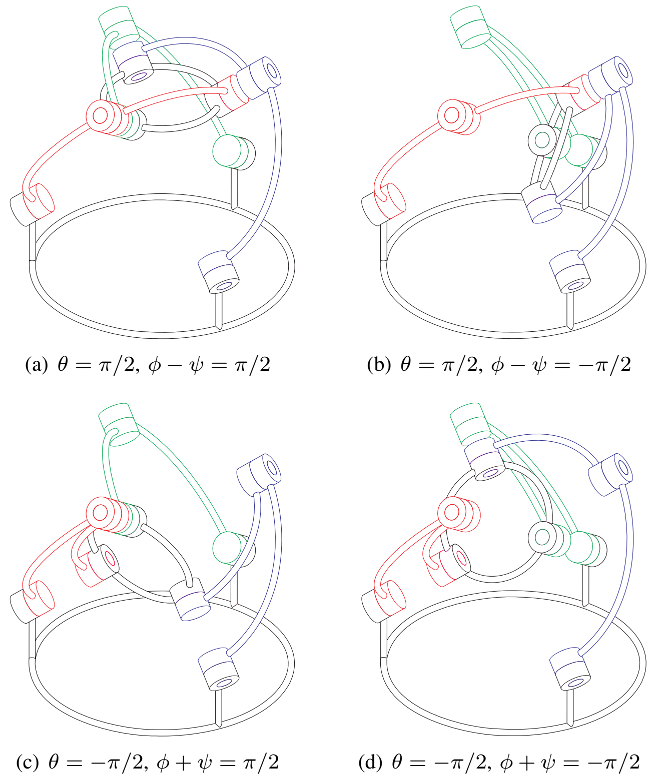
$$\phi = \theta_3 \quad \text{and} \quad \phi = \theta_3 \pm \pi. \tag{14}$$

However, in the *ZYX* Euler-angle convention, the triplets $\{\phi, \theta, \psi\}$ and $\{\phi \pm \pi, -\theta \pm \pi, \psi \pm \pi\}$ both correspond to the same orientation. This means that the above two solutions will lead to the same orientation of the mobile platform. Hence, only the first solution will be used in this paper.

Substituting $\phi = \theta_3$ in (9b-c), a new system is obtained:

$$p_1 \cos\psi + p_2 \sin\psi = 0, \tag{15a}$$

$$p_3 \cos\psi + p_4 \sin\psi = 0, \tag{15b}$$



(a) $\theta = \pi/2$, $\phi - \psi = \pi/2$ (b) $\theta = \pi/2$, $\phi - \psi = -\pi/2$ (c) $\theta = -\pi/2$, $\phi + \psi = \pi/2$ (d) $\theta = -\pi/2$, $\phi + \psi = -\pi/2$

Fig. 2. The four trivial solutions to the direct kinematic problem valid for any set of active-joint variables but shown with $\theta_1 = 0$, $\theta_2 = 0$ and $\theta_3 = 0$.

where

$$p_1 = \sin\theta_1 \cos\theta_3, \tag{15c}$$

$$p_2 = \sin\theta_1 \sin\theta \sin\theta_3 - \cos\theta \cos\theta_1, \tag{15d}$$

$$p_3 = \cos\theta_2 \sin\theta \cos\theta_3 - \cos\theta \sin\theta_2, \tag{15e}$$

$$p_4 = \cos\theta_2 \sin\theta_3. \tag{15f}$$

Since the terms $\cos\psi$ and $\sin\psi$ cannot vanish simultaneously, (15a-b) lead to

$$p_1 p_4 - p_2 p_3 = 0. \tag{16}$$

Hence, substituting (15c-f) into the above equation gives

$$\cos\theta(q_1 \cos\theta + q_2 \sin\theta) = 0, \tag{17a}$$

where

$$q_1 = \sin\theta_1 \cos\theta_2 \cos\theta_3 \sin\theta_3 - \cos\theta_1 \sin\theta_2, \tag{17b}$$

$$q_2 = \sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3. \tag{17c}$$

Equation (17a) leads to two possibilities:

$$\cos\theta = 0 \quad \text{and} \tag{18a}$$

$$q_1 \cos\theta + q_2 \sin\theta = 0. \tag{18b}$$

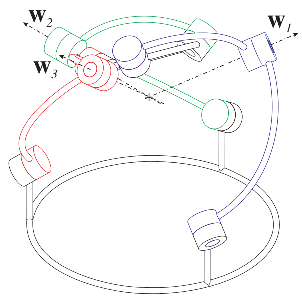
Equation (18a) is the same as (10a) and will therefore be discarded. Equation (18b) gives two solutions in $(-\pi, \pi]$:

$$\theta = \tan^{-1}(-q_1/q_2) + k\pi \text{ with } k = 0, 1. \tag{19}$$

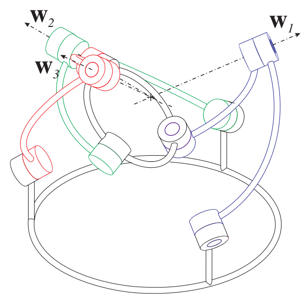
Any one of (15a-b) can now be used to find ψ, e.g.:

$$\psi = \tan^{-1}(-p_1/p_2) + k\pi \text{ with } k = 0, 1, \tag{20}$$

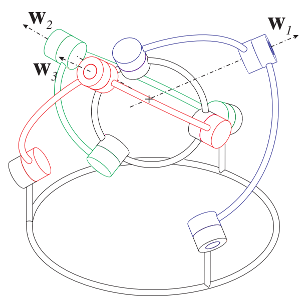
which gives two values for ψ in $(-\pi, \pi]$ for each θ.



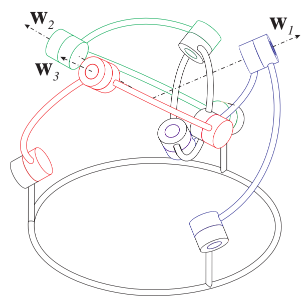
(a) $\phi = 0.100$, $\theta = -0.672$, $\psi = -0.383$



(b) $\phi = 0.100$, $\theta = -0.672$, $\psi = 2.759$



(c) $\phi = 0.100$, $\theta = 2.470$, $\psi = 0.383$



(d) $\phi = 0.100$, $\theta = 2.470$, $\psi = 3.525$

Fig. 3. The four nontrivial solutions to the direct kinematic problem of the *Agile Eye* for $\theta_1 = -0.3, \theta_2 = -0.7$ and $\theta_3 = 0.1$.

Solution 1	ϕ	θ	ψ
Solution 2	ϕ	θ	$\psi + \pi$
Solution 3	ϕ	$\theta + \pi$	$-\psi$
Solution 4	ϕ	$\theta + \pi$	$-\psi + \pi$

TABLE I
THE FOUR NONTRIVIAL SOLUTIONS TO THE DIRECT KINEMATIC PROBLEM

Table I summarizes the solutions to the direct kinematic problem, where a number is arbitrarily assigned to each solution. An example of four nontrivial solutions to the direct kinematic problem of the *Agile Eye* is given in Fig. 3 for the active-joint variables $\theta_1 = -0.3, \theta_2 = -0.7$ and $\theta_3 = 0.1$.

Later, a method will be presented to identify these nontrivial solutions based on the working mode. For now, note by solely observing Fig. 3 that these four nontrivial solutions are obtained from each other by rotating the mobile platform about a platform joint axis at 180 degrees (the same is true for the four trivial solutions). Thus, if for a given set of active-joint variables, there is one nontrivial solution, then there are (at least) three other nontrivial solutions. The important question whether there are always four nontrivial solutions was not answered in [11] and will now be given special attention.

C. Degenerate Cases

The first and second set of solutions will become the same when (18b) degenerates and $q_2 = q_1 = 0$. It can be shown that $q_2 = q_1 = 0$ if, and only if, $\sin\theta_2 = 0$ and $\cos\theta_3 = 0$, or $\sin\theta_3 = 0$ and $\cos\theta_1 = 0$. In that case, θ can be anything, which is one of the self-motions of the platform. If $q_2 = 0$ but $q_1 \neq 0$, (18b) becomes identical with (18a), meaning that $\theta = \pm\pi/2$. Substituting $\theta = \pm\pi/2$ into (15a-b) yields

$$\sin\theta_1 \cos(\phi \mp \psi) = 0 \text{ and } \cos\theta_2 \cos(\phi \mp \psi) = 0. \quad (21)$$

If $\sin\theta_1 = 0$ and $\cos\theta_2 = 0$, $(\phi \mp \psi)$ is arbitrary, meaning that the platform can undergo a self-motion. If, however, these two conditions are not satisfied, then $\cos(\phi \mp \psi) = 0$, meaning that the only direct kinematic solutions are the trivial ones.

Thus, in summary, the *Agile Eye* will have only the four trivial solutions to its direct kinematic problem if and only if

$$q_2 = \sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3 = 0, \quad (22)$$

but neither of the following three pairs of conditions is true:

$$\sin\theta_2 = 0 \text{ and } \cos\theta_3 = 0, \text{ or} \quad (23a)$$

$$\sin\theta_3 = 0 \text{ and } \cos\theta_1 = 0, \text{ or} \quad (23b)$$

$$\sin\theta_1 = 0 \text{ and } \cos\theta_2 = 0. \quad (23c)$$

Taking the first pair yields $q_2 = 0$ and $q_1 = 0$, meaning that θ can take any value. This means that there is a self-motion — even if all actuators are fixed, the platform can freely move. Substituting $\theta_2 = 0$ or $\theta_2 = \pi$, and $\theta_3 = \pm\pi/2$ into (15a-b), yields $\sin\psi = 0$. In addition, $\cos\phi = \cos\theta_3 = 0$. Similarly, for the second pair of conditions, it can be proved that θ can have any value, while $\cos\psi = 0$ and $\sin\phi = 0$. Finally, taking the third pair of conditions, $q_2 = 0$ but $q_1 \neq 0$, (17a) implies that $\cos\theta = 0$. If this pair of conditions is substituted into (15a-b), it is reached to the conclusion that $\sin(\phi - \psi)$ can be anything in the case of $\theta = \pi/2$ or that $\sin(\phi + \psi)$ can be anything in the case of $\theta = -\pi/2$.

Thus, it can be easily shown that (23a–c) correspond to six self-motions represented by the following rotation matrices (where all angles are arbitrary):

$$\mathbf{R}_{SM1a} = \begin{bmatrix} 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad (24a)$$

$$\mathbf{R}_{SM1b} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & -\cos\theta \end{bmatrix}, \quad (24b)$$

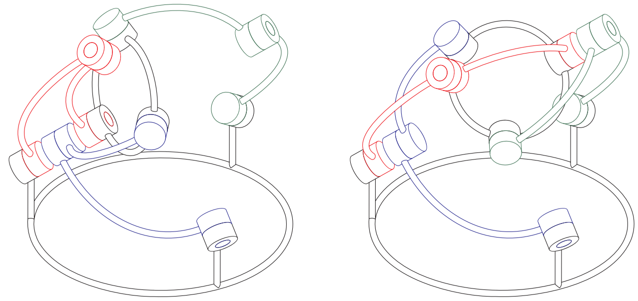
$$\mathbf{R}_{SM2a} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & -1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}, \quad (24c)$$

$$\mathbf{R}_{SM2b} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & -\cos\theta & 0 \end{bmatrix}, \quad (24d)$$

$$\mathbf{R}_{SM3a} = \begin{bmatrix} 0 & -\sin(\phi-\psi) & \cos(\phi-\psi) \\ 0 & \cos(\phi-\psi) & \sin(\phi-\psi) \\ -1 & 0 & 0 \end{bmatrix}, \quad (24e)$$

$$\mathbf{R}_{SM3a} = \begin{bmatrix} 0 & -\sin(\phi+\psi) & -\cos(\phi+\psi) \\ 0 & \cos(\phi+\psi) & -\sin(\phi+\psi) \\ 1 & 0 & 0 \end{bmatrix}. \quad (24f)$$

Now, note that each of the pairs of conditions imposes a constraint on two of the active-joint variables, while the third one can take any value, without influencing the orientation



(a) (b)

Fig. 4. The two self-motions of the mobile platform where leg 3 is singular.

of the mobile platform. This means, that in the above self-motions, there is a leg in singularity. In fact, the self-motion of the platform is about the axis of the base joint of the leg in singularity. The above self-motions are divided into pairs, where in each pair, one of the motions correspond to a fully extended leg (the ones with the b index), while the other to a fully folded one. Figure 4 shows the two self-motions (SM3a and SM3b) for which leg 3 is singular, corresponding to the pair of conditions of (23c). In this figure, legs 1 and 2 are each shown in one of the possible two configurations per leg.

As will be seen in the next section, all singular configurations were found in this section by purely studying the degeneracies of the direct kinematics of the *Agile Eye*.

V. Singularity Analysis

The relationship between the active-joint rates, $\dot{\boldsymbol{\theta}}$, and the angular velocity of the mobile platform, $\boldsymbol{\omega}$, can be written as:

$$\mathbf{A}\boldsymbol{\omega} + \mathbf{B}\dot{\boldsymbol{\theta}} = 0 \tag{25}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the following Jacobian matrices:

$$\mathbf{A} = \begin{bmatrix} (\mathbf{w}_1 \times \mathbf{v}_1)^T \\ (\mathbf{w}_2 \times \mathbf{v}_2)^T \\ (\mathbf{w}_3 \times \mathbf{v}_3)^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T \end{bmatrix}, \tag{26a}$$

$$\mathbf{B} = \begin{bmatrix} (\mathbf{w}_1 \times \mathbf{v}_1)^T \mathbf{u}_1 & 0 & 0 \\ 0 & (\mathbf{w}_2 \times \mathbf{v}_2)^T \mathbf{u}_2 & 0 \\ 0 & 0 & (\mathbf{w}_3 \times \mathbf{v}_3)^T \mathbf{u}_3 \end{bmatrix}. \tag{26b}$$

Type 2 singularities are characterized by studying matrix $\mathbf{A}$ and occur whenever the three vectors $\boldsymbol{\alpha}_i$ are coplanar or collinear. For the *Agile Eye*, these vectors cannot be collinear. Thus, when these three vectors are coplanar, the platform can rotate (infinitesimally or finitely) about the axis passing through the center O and normal to the plane of the vectors.

Substituting the nontrivial solution set for the direct kinematics of the *Agile Eye* into the determinant of $\mathbf{A}$ and simplifying, it is obtained that the expression in the active-joint space for Type 2 singularities is:

$$\det(\mathbf{A}) = \sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3 = 0. \tag{27}$$

Substituting the trivial solution set yields the same expression but with an opposite sign. Note that the determinant of $\mathbf{A}$ is the same for all assembly modes. Indeed, from the geometric interpretation of the four assembly modes, it can be seen that between any two assembly modes, two pairs of vectors $\boldsymbol{\alpha}_i$ have opposite directions and the third pair is the same.

Type 1 singularities are characterized by studying matrix $\mathbf{B}$ and occur whenever a leg is fully extended or folded. For a general 3-*R̲RR* spherical parallel mechanism with legs of other than 90 degrees, Type 1 singularities are two-dimensional entities. In other words, for such a general parallel wrist, when a single leg is singular, the platform remains with two degrees of freedom. In the *Agile Eye*, however, when a leg is singular, the axes of the base and platform joints coincide and the mobile platform has a single degree of freedom, whereas the leg can freely rotate without affecting the orientation of the mobile platform. Therefore, Type 1 singularities of the *Agile Eye* are only six curves in the orientation space.

Substituting the nontrivial solution set for the direct kinematics of the *Agile Eye* into $\mathbf{B}$ and simplifying yields the following three expressions in the active-joint space, corresponding to Type 1 singularities occurring in leg 1, 2, and 3, respectively:

$$B_{11} = \pm \frac{\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3}{\sqrt{1-\cos^2\theta_3 \sin^2\theta_1}\sqrt{1-\cos^2\theta_1 \sin^2\theta_2}} = 0, \tag{28}$$

$$B_{22} = \pm \frac{\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3}{\sqrt{1-\cos^2\theta_2 \sin^2\theta_3}\sqrt{1-\cos^2\theta_1 \sin^2\theta_2}} = 0, \tag{29}$$

$$B_{33} = \pm \frac{\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \cos\theta_3}{\sqrt{1-\cos^2\theta_2 \sin^2\theta_3}\sqrt{1-\cos^2\theta_3 \sin^2\theta_1}} = 0, \tag{30}$$

where B_{ii} is the i-th diagonal element of $\mathbf{B}$, and the plus-minus sign depends on which of the four nontrivial direct solutions is used, i.e., on the assembly mode.

Substituting the trivial solution set for the direct kinematics of the *Agile Eye* into $\mathbf{B}$ and simplifying yields as expected:

$$B_{11} = B_{22} = B_{33} = 0. \tag{31}$$

Indeed, at the trivial orientations, all three legs are singular.

From (28–30), it follows that if a configuration corresponds to a Type 2 singularity, then it should inevitably correspond to a Type 1 singularity too. However, the opposite is not necessarily true. In other words, if there is a Type 1 singularity, the *Agile Eye* is not necessarily at a Type 2 singularity too. Indeed, investigating the four orientations shown in Fig. 2, it can be seen that the legs can be orientated in such a way that the vectors normal to the last two joint axes in each leg are not coplanar. Such a configuration is called a lockup configuration, since the mobile platform is completely restrained even if the actuators are removed. For these four orientations, (28–30) are replaced by (31).

Finally, it should be verified what happens, when the denominators of the expressions in (28–30) are zeroed. This basically occurs when from the reference configuration, a leg is turned at 90° (in either direction), and possibly another one is turned at 180°. It can be verified that at such a configuration, a Type 1 singularity occurs. Indeed, the numerator of the above three equations also becomes zero.

In conclusion, the *Agile Eye* has the following three families of singular configurations:

- **Six self-motions** of the mobile platform described by (24a–f), in SO(3), and by (23a–c), in the active-joint space. For each self-motion, one of the legs is singular. For the active-joints corresponding to each of (23a–c), the four (nontrivial) assembly modes degenerate in pairs into the two corresponding self-motions.
- **Infinitely many infinitesimal motions** of the mobile platform at the four trivial orientations when the active-joints satisfy (27) but neither of (23a–c). Each of the four trivial orientations belongs to the sets of orientations corresponding to three self-motions. For the active joint variables corresponding to these singular configurations, the only solutions to the direct kinematic problem are the four trivial ones.
- **Lockup configuration** described in SO(3) by each of the four trivial orientations and in the active-joint space space by anything but (27). At these Type 1 singular configurations, the mobile platform cannot move even under external force. For the active joint variables corresponding to these singular configurations, the direct kinematic problem has eight solutions.

VI. Working Modes and Assembly Modes

As was mentioned before, a particularity of the *Agile Eye* is that the determinant of matrix $\mathbf{A}$, (27), is a function of the active-joint variables only and has the same value for all four nontrivial assembly modes. Equation (27) represents a surface that divides the active-joint space into two domains where $\det(\mathbf{A})$ is either positive or negative. A connectivity analysis was made on these two domains to prove this property.

The images of these two domains in the workspace yield eight identical domains (the whole orientation space without the singularity curves), each one being associated with one of the eight working modes. It can been seen geometrically, or proved algebraically, that each of the four nontrivial assembly modes corresponds to a different working mode. Therefore, changing an assembly mode inevitably requires a singularity to be crossed. Hence, the *Agile Eye* is not cuspidal, meaning that as long as it does not cross a singularity, it remains in a single working mode and in a single assembly mode.

The eight working modes are divided into two groups. Depending on the sign of $\det(\mathbf{A})$, each of the four assembly modes corresponds to a working mode from one of these groups. In Table II, it is assumed that for the first solution of the direct kinematics, the sign of B_{ii} given in (28–30) is negative. For the other solutions, $\{\psi, \theta, \phi\}$ are replaced by the value given in Table I and the influence on the sign of B_{ii} is shown using simple trigonometric properties. Recall that $|B_{ii}|$ does not change when an assembly mode is changed.

Figure 5 summarizes the behavior of the direct kinematic problem where a solution can be chosen according to the sign of $\mathbf{B}_{ii}$ and the sign of $\det(\mathbf{A})$. Similarly, when the inverse kinematic model is solved, the working mode is easily characterized by the sign of $\mathbf{B}_{ii}$.

Sign of	B_{11}	B_{22}	B_{33}	$\det(\mathbf{B})$
Solution 1	$-$	$-$	$-$	$-$
Solution 2	$+$	$+$	$-$	$-$
Solution 3	$-$	$+$	$+$	$-$
Solution 4	$+$	$-$	$+$	$-$

TABLE II

THE SIGNS OF $\mathbf{B}_{ii}$ FOR A GIVEN SET OF ACTIVE-JOINT VARIABLES

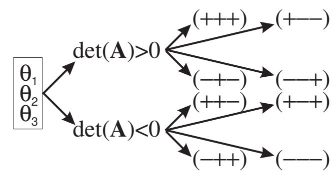


Fig. 5. The eight working modes characterized by the assembly mode.

VII. Conclusions

An in-depth kinematic analysis of a special spherical parallel wrist, called the *Agile Eye*, was done, pinpointing some important facts that were previously overlooked. It was demonstrated that the workspace of the *Agile Eye* is unlimited and flawed only by six singularity curves (rather than surfaces). Furthermore, these curves were shown to correspond to self-motions of the mobile platform. It was also proved that the four assembly modes of the *Agile Eye* are directly related to the eight working modes and the sign of the determinant of one of the Jacobian matrices. It was shown that as long as the *Agile Eye* does not cross its singularity curves, it remains in a single working mode and in a single assembly mode.

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