Proceedings of the 1999 IEEE/ASME
International Conference on Advanced Intelligent Mechatronics
September 19-23, 1999 • Atlanta, USA

A Simple New Closed-Form Solution of the Direct Kinematics of Parallel Manipulators Using Three Linear Extra Sensors

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Abstract— **This paper presents a simple new closed-form solution of the direct kinematics problem of general parallel manipulators, by using three linear extra sensors. The sensors are disposed at a most general way, connecting the planar base and the planar mobile platform at distinct points. The basic idea is to use the coordinates of the three distinct attachment points of the extra sensors on the mobile platform to represent the posture of the mobile platform. Thus, the extra sensory data enable to reduce the problem to the solution of a system of 6 linear equations in 6 of the 9 generalized coordinates. The other 3 coordinates are obtained directly from the extra sensory data.**

Keywords— **parallel manipulator, direct kinematics, linear extra sensors.**

I. Introduction

A 6-DOF parallel manipulator consists of a *mobile platform* connected by six links to a *base* through respectively spherical and universal joints. Most commonly, the base joints are fixed on the base while the links are of variable length. Changing the link lengths controls the position and orientation (the *posture*) of the mobile platform. The main advantages of parallel robots are the high load/weight ratio and the high positioning accuracy due to their inherent rigidity.

While the inverse kinematics of parallel robots is nearly straightforward, the *direct kinematics problem* (DKP) is a very difficult one and has initiated much research during the past decade. The DKP relates to the determination of the mobile platform posture for a given set of control inputs. It involves the solution of a system of nonlinear coupled algebraic equations in the variables describing the mobile platform posture (the *generalized coordinates*) and has many solutions, referred to as *assembly modes.*

The complexity of the DKP has urged the creation of simplified architectures, identified by some special arrangement such as coalescence or collinearity of some or all of the base and/or platform joints (e.g. [1]). Even though such architectures can have as few as 16 closed-form solutions to the DKP [1], they require a complicated mechanical design and often yield a degraded robot performance. Hence, it is necessary to study the DKP for the *most general parallel manipulator*, with 6 distinct base and platform joints and no constraints on their location.

For the *nearly general parallel manipulators* (planar base and platform), Lazard [2] has shown that the DKP can have up to 40 solutions. Zhang and Song [3] and Wen and Liang [4] arrive at a *univariate polynomial* (UP) of degree 20 whose solutions lead to 40 assembly modes. For the most general parallel manipulators, Raghavan [5] has demonstrated with numerical examples that the maximal number of solutions is still 40. The same has been proved in [6]-[9]. Husty [10] was the first to reach to a UP of degree 40. The complete solution of the DKP can bring a better insight into the robot kinematics. In control system design or 3D coordinate measurement, however, we are interested only in the "*true*" solution—the assembly mode in which the measurement is taken.

Two common approaches to find directly the "true" solution are: (i) to use an iterative numerical procedure or (ii) to use extra sensors. For the numerical method, as the motion is continuous, an initial estimate at each step is the posture at the previous step. The convergence problem is solved mainly by choosing a very small time step, which requires fast computation and is not suitable for real-time applications unless expensive computer hardware is used.

For the extra sensors, the redundant data may be used to determine uniquely the solution to the DKP and possibly to provide a good initial estimate for an iterative method that has to be reset often due to error accumulation. Other advantages of the extra sensors

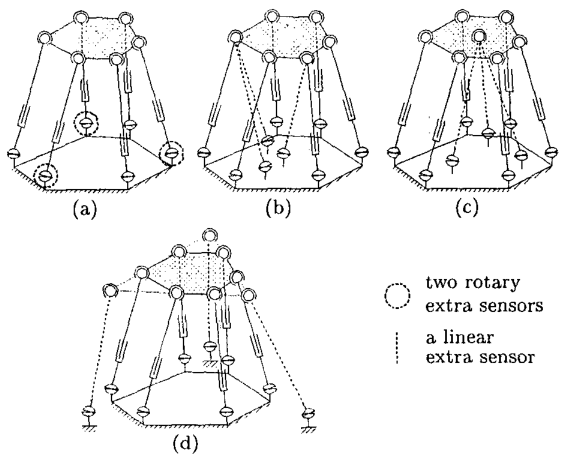

Fig. 1. Type and disposition of extra sensors leading to a unique solution to the DKP: (a) rotary sensors attached at the axes of the base universal joints, (b)-(c) linear sensors attached in a tripod configuration, and (d) the proposed most general attachment of linear sensors.

are that they can be used for robot self-calibration [11] and can ensure the robot's reliability in case of a sensor failure [12].

From a theoretical point of view, the choice of type and disposition of extra sensors is aimed only at obtaining the unique solution to the DKP with a minimum number of sensors. From a practical point of view, however, the latter is only a necessity while an optimum choice is characterized by minimum overall cost, minimum risk of link-sensor interference, minimum error on the obtained solution, and many other features.

Two types of extra sensors have been used: *rotary* and *linear*. The first type is placed at both axes of a base universal joint, measuring the link direction and allowing to obtain the position of the corresponding platform joint (Fig. 1a). Various sensor dispositions have been proposed [12]-[14]. At least 4 rotary extra sensors are needed for obtaining a unique solution. The problems associated with their implementation are error build-up and difficulties in their attachment.

The second type of sensors, the linear sensors, connect the base and the mobile platform and are used in two configurations. In the first one, two sensors are attached to a mobile platform joint (Fig. 1b). Together with the link, they form a *tripod* geometry, which enables to find the position of that joint [15]. Hence, this arrangement is kinematically equivalent to the use of 2 rotary sensors at a base universal joint. In the second configuration, 3 linear extra sensors are connected to the center of the platform (Fig. 1c), thus, allowing to calculate the position of the mobile platform [15], [16]. Its orientation is then determined from the six link lengths by solving a system of 6 linear equations. Nair [16] has proved that for only 2 linear extra sensors there exist up to 8 explicit solutions, so at least 3 linear sensors are needed for obtaining a unique solution. Major problems of these configurations are an increased risk of link interference and difficulty in constructing the triple spherical joint.

In the present work, we propose a method for finding a simple closed-form solution of the DKP of nearly general parallel manipulators. The method is based on the use of three linear extra sensors connecting the planar base and the planar mobile platform at *distinct* points—an arrangement which is the easiest to realize (Fig. 1d). A proper location of the extra sensors' attachment points may eliminate any interference with the manipulator links.

In our method, we use the coordinates of the three attachment points of the extra sensors, i.e. 9 generalized coordinates. Thus, the DKP can be represented as 6 quadratic equations coming from the link lengths, and 3 quadratic equations coming from the fixed distances between the three attachment points. The 3 linear sensors introduce 3 new quadratic equations. Together with the other 3 constraint equations (the constant distances between the sensor attachment points), they are used to linearize the first 6 equations. Due to the coplanarity constraint, the linearized system contains only the x and y generalized coordinates, which therefore can be found explicitly. The z generalized coordinates can then be solved from the 3 equations coming from the extra sensors.

In the next section, we introduce the formulation of the direct kinematics equations and the manipulations that lead to the linearized system. Section III presents a discussion on the implementation of the proposed solution, and section IV enumerates the possible directions of further study. Conclusions are given in the last section V.

II. Direct Kinematics Formulation and Solution

The complexity of the DKP depends greatly on the choice of generalized coordinates (see [7]). The 6-DOF platform posture may be represented by using 3 position coordinates and 3 orientation angles. However, these generalized coordinates lead to complicated trigonometric equations of high degree that are difficult to manipulate. Another common approach is to parameterize the rotation matrix in terms of direction cosines or dual quaternions [8], [9], [16]. This approach is useful when the base joints are coplanar but leads to complicated expressions in the non-coplanar case.

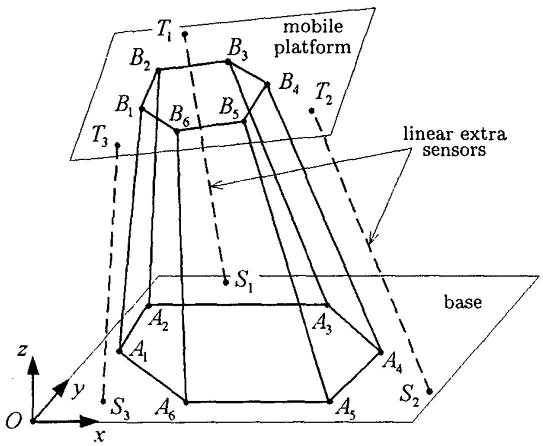

Fig. 2. A nearly general 6-6 parallel manipulator (with planar base and planar mobile platform) with three linear extra sensors.

In this paper, the posture is described by the 9 coordinates of 3 non-collinear points on the platform, subjected to the constraint that their relative distances are fixed. This choice of generalized coordinates leads to a system of 6 quadratic and 3 linear equations [2], [4], [17] in the case of a planar mobile platform.

Consider a *nearly general 6-6 parallel manipulator* with a planar base and a planar mobile platform as shown in Fig. 2. The base joints are denoted by A_i and the platform joints by B_i ($i = 1, \ldots, 6$). An absolute reference frame is chosen with center O fixed to the base so that its x- and y-axes lie in the plane of the joints A_i. The coordinates of points A_i are x_{A_i}, y_{A_i}, and z_{A_i}, where $z_{A_i} = 0$. The generalized coordinates describing the platform posture with respect to the base are chosen to be the 9 coordinates of 3 points T_i on the mobile platform, namely x_i, y_i, and z_i ($i = 1, 2, 3$). The points are selected in the plane of points B_i and the distance between each two of them is equal to k. Note that it is not necessary for these points to be equidistant.

The coordinates of points B_i are expressed in terms of the generalized coordinates (Fig. 3) as

$$\mathbf{OB}_i = \begin{bmatrix} k_{i,1}x_1 + k_{i,2}x_2 + k_{i,3}x_3 \\ k_{i,1}y_1 + k_{i,2}y_2 + k_{i,3}y_3 \\ k_{i,1}z_1 + k_{i,2}z_2 + k_{i,3}z_3 \end{bmatrix}, \; i = 1, \ldots, 6 \quad (1)$$

where $k_{i,1} + k_{i,2} + k_{i,3} = 1$. Indeed, this is true because

$$\begin{aligned} \mathbf{OB}_i &= \mathbf{OT}_3 + k_{i,1}\mathbf{T}_3\mathbf{T}_1 + k_{i,2}\mathbf{T}_3\mathbf{T}_2 \\ &= k_{i,1}\mathbf{OT}_1 + k_{i,2}\mathbf{OT}_2 + (1 - k_{i,1} - k_{i,2})\mathbf{OT}_3. \quad (2) \end{aligned}$$

Note, however, that when points B_i are not in the plane of the three points T_i, the expressions for their coordinates are much more complicated.

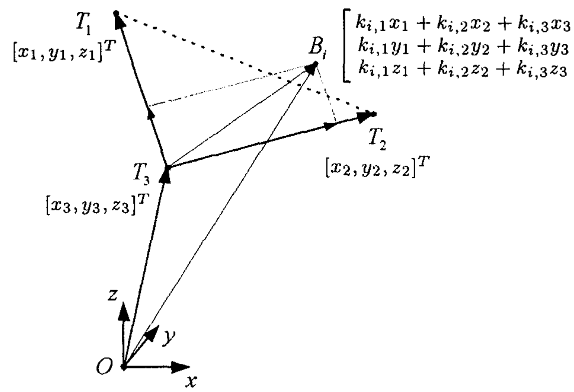

Fig. 3. Representing the coordinates of the platform joints in the generalized coordinates.

For a given set of control variables, the direct kinematics problem may be set as a system of 6 constraint equations associated with the lengths of the links (Subsystem I) and 3 constraint equations on the generalized coordinates (Subsystem II) in 9 unknowns:

Subsystem I:

$$\begin{aligned} \Phi_i = & (k_{i,1}x_1 + k_{i,2}x_2 + k_{i,3}x_3 - x_{A_i})^2 \\ & + (k_{i,1}y_1 + k_{i,2}y_2 + k_{i,3}y_3 - y_{A_i})^2 \\ & + (k_{i,1}z_1 + k_{i,2}z_2 + k_{i,3}z_3)^2 - \ell_i^2 = 0, i = 1, \ldots, 6 \quad (3) \end{aligned}$$

Subsystem II:

$$\begin{aligned} \Phi_7 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - k^2 = 0 \\ \Phi_8 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 - k^2 = 0 \\ \Phi_9 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 - k^2 = 0, \quad (4) \end{aligned}$$

where ℓ_i is the length of link i ($i = 1, \ldots, 6$).

Subsystems I and II can be reduced to a system of 6 linear and 3 quadratic equations in the 9 generalized coordinates [17]. This system of equations will have a maximum of 64 solutions by Bezout's theorem, and is still too complex to be solved. Therefore, this paper proposes a more efficient treatment of the sets of equations (3) and (4) by utilizing 3 linear extra sensors.

The linear extra sensors (Fig. 2) are measuring the distances between three distinct points S_i ($i = 1, 2, 3$) on the base, with absolute coordinates x_{S_i}, y_{S_i}, and 0, and respectively the points T_i ($i = 1, 2, 3$). Thus, three new equations are generated as:

Subsystem III:

$$\begin{aligned} \Phi_{10} &= (x_1 - x_{S_1})^2 + (y_1 - y_{S_1})^2 + z_1^2 - s_1^2 = 0 \\ \Phi_{11} &= (x_2 - x_{S_2})^2 + (y_2 - y_{S_2})^2 + z_2^2 - s_2^2 = 0 \\ \Phi_{12} &= (x_3 - x_{S_3})^2 + (y_3 - y_{S_3})^2 + z_3^2 - s_3^2 = 0, \quad (5) \end{aligned}$$

where s_i is the length of extra sensor i ($i = 1, 2, 3$).

The extra sensory data represented by Subsystem III can now be used together with Subsystem II to linearize Subsystem I:

Linearized Subsystem I:

$$\begin{aligned}\Phi_i' &= \Phi_i - k_{i,1}^2\Phi_{10} - k_{i,2}^2\Phi_{11} - k_{i,3}^2\Phi_{12} \\ &+k_{i,1}k_{i,2}(\Phi_7 - \Phi_{10} - \Phi_{11}) \\ &+k_{i,2}k_{i,3}(\Phi_8 - \Phi_{11} - \Phi_{12}) \\ &+k_{i,3}k_{i,1}(\Phi_9 - \Phi_{12} - \Phi_{10}), \quad i = 1, \ldots, 6. \quad (6)\end{aligned}$$

Now, if points S_i $(i = 1, 2, 3)$ and points A_i $(i = 1, \ldots, 6)$ are not coplanar, then the Linearized Subsystem I will consist of 6 linear equations in the 9 generalized coordinates. In such a case, 3 of the 9 generalized coordinates can be chosen as independent, and the other 6 generalized coordinates can be expressed linearly in terms of the independent 3 by solving the set of equations (6). These expressions can be further substituted into Subsystems II and III to result in two systems of quadratic equations in the three independent generalized coordinates. Each of these systems can have up to 8 solutions (real and complex), which can be readily obtained. A comparison of the 8 solutions of the first system with the 8 solutions of the second subsystem will reveal the true solution to the DKP, but it remains unclear whether that solution will be unique [19].

If points S_i $(i = 1, 2, 3)$ and points A_i $(i = 1, \ldots, 6)$ are coplanar, as we have already assumed, then the Linearized Subsystem I contains only 6 of the 9 generalized coordinates, namely x_1, y_1, x_2, y_2, x_3, and y_3. Thus, this system can be written as

$$\mathbf{M}\mathbf{q} = \mathbf{L}, \tag{7}$$

where $\mathbf{M}$ is a 6×6 coefficient matrix whose row i $(i = 1, \ldots, 6)$ is given by

$$\begin{aligned}\mathbf{M}_i = 2\big[&k_{i,1}(x_{S_1} - x_{A_i}),\ k_{i,1}(x_{S_1} - x_{A_i}), \\ &k_{i,2}(x_{S_2} - x_{A_i}),\ k_{i,2}(x_{S_2} - x_{A_i}), \\ &k_{i,3}(x_{S_3} - x_{A_i}),\ k_{i,3}(x_{S_3} - x_{A_i})\big],\end{aligned}$$

$\mathbf{L}$ is a 6×1 vector whose element i $(i = 1, \ldots, 6)$ is given by

$$\begin{aligned}L_i &= (\ell_i^2 - x_{A_i}^2 - y_{A_i}^2) - k_{i,1}(s_1^2 - x_{S_1}^2 - y_{S_1}^2) \\ &-k_{i,2}(s_2^2 - x_{S_2}^2 - y_{S_2}^2) - k_{i,3}(s_3^2 - x_{S_3}^2 - y_{S_3}^2) \\ &+k^2(k_{i,1}k_{i,2} + k_{i,2}k_{i,3} + k_{i,3}k_{i,1}),\end{aligned}$$

and $\mathbf{q}$ is the 6×1 *reduced vector of generalized coordinates*

$$\mathbf{q} = \left[x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3\right]^T.$$

Note that matrix $\mathbf{M}$ is purely dependent on the geometry of the mobile platform and the arrangement of the sensor base attachment points and base joints, and is hence constant for a given parallel manipulator with extensible links. If this matrix is of full rank, then we can explicitly solve for the reduced vector of generalized coordinates as:

$$\mathbf{q} = \mathbf{M}^{-1}\mathbf{L}. \tag{8}$$

After solving eq. (8), the generalized coordinates x_1, y_1, x_2, y_2, x_3, and y_3 can be, for example, substituted into eqs. (5) (Subsystem III) to compute directly the values for the other 3 generalized coordinates z_1, z_2, and z_3. Since in practice we need only the positive values of z_1, z_2, and z_3, this concludes the determination of the unique solution of the DKP.

Assuming that for a given design, matrix $\mathbf{M}$ is not singular, then the proposed method for solving the DKP will work for any configuration of the parallel manipulator, including singular configurations. Note that no method for solving the DKP that does not use extra sensors works at singular configurations.

III. Implementation Issues

Basically, in the case of medium-cost parallel manipulators, two options exist for the type of linear extra sensors: *Linear Variable Differential Transformers (LVDTs)* and *Cable Extension Transducers (CETs)*, known also as string pots. The first type typically allows a low measurement range (0.5 m), its price is high, it requires support electronics, and its installation is difficult, requiring the use of at least universal joints and an additional casing. Further, in case of link-sensor interference, the sensor is surely damaged.

The other option, the CETs, produce electrical signals proportional to the travel of their extension cables. The cable end is attached to the moveable object while the device itself is fixed to a stationary part. As the object moves, the cable extends or retracts. The linear displacement is then converted to angular displacement with the cable being wound onto a cylindrical spool. A rotary sensor (shaft encoder or potentiometer) measures the spool rotation. The tension of the cable is guaranteed by a spring connected to the spool. Using a cable guide, the cable is allowed to move within a 20° cone, making it suitable for three-dimensional movement applications such as that of the mobile platform. Manufacturers of such sensors are Celesco Transducer Products Inc., SpaceAge Control Inc., Carlen Controls Inc., and several others.

CETs provide a long range (0.04∼40 m), typical accuracy of 0.2% of full scale (FS) in case of potentiometers and 0.02% FS in case of shaft encoders, and typical repeatability of 0.02% FS. The maximum allowable cable velocity is about 7.2 m/s and the maximum cable acceleration is about 200 m/s^2. The characteristics of a typical CET make it appropriate for the discussed application.

IV. Discussions and Further Study

The optimum implementation of the linear extra sensors is to be determined on the basis of several criteria: (i) singularity of matrix $\mathbf{M}$, (ii) sensor misalignment range, and (iii) link-sensor interference.

Firstly, matrix $\mathbf{M}$ should not be singular. As we already mentioned, in the case of parallel manipulators with extensible links, this matrix is constant and depends only on the geometry of the manipulator's base and mobile platform. If, however, the parallel manipulator is with constant links, e.g. the "Hexaglide" robot at ETH Zurich [20], then both matrix $\mathbf{M}$ and vector $\mathbf{L}$ will be configuration dependent, and thus, a careful analysis of the rank of matrix $\mathbf{M}$ will be required.

Secondly, as we saw from the previous section, each extra sensor should stay in a 20° cone throughout the full range of motion of the mobile platform. For that purpose, the maximal workspace at points T_i ($i = 1, 2, 3$) should be computed. Finally, it is clear that the farther the points S_i ($i = 1, 2, 3$) from points A_i ($i = 1, \ldots, 6$), the lower the risk of link interference and the lower the misalignment angle for the cable.

Since the proposed method for solving the DKP involves only a multiplication of a constant coefficient matrix ($\mathbf{M}^{-1}$) with a configuration-dependent vector ($\mathbf{L}$) and three square-root operations, it is particularly suited for real-time implementation. It can also be used in conjunction with a numerical method, such as the Newton-Raphson's one, to provide it with an initial estimate in order to compute a solution to the DKP that is not dependent on the extra sensors' accuracy. This solution scheme is also very suitable for use in robot self calibration due to the simple form of the expressions defining the unique solution that facilitates the error analysis.

V. Conclusion

A simple exact method for solving the direct kinematics problem of parallel manipulators with planar base and mobile platform was proposed. The method is based on the use of three linear extra sensors to provide additional information that is used to linearize the initial six constraint equations that come from the link lengths. The simplicity of the method is derived from the use of the coordinates of the three attachment points of the extra sensors to the base to serve as generalized coordinates.

Acknowledgement

The work has been partially supported by Kumho Tire & Co, Inc.

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