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New Mixed Method for Unsteady Aerodynamic Force Approximations for Aeroservoelasticity Studies

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Introduction

THE unsteady aerodynamic forces acting on a business aircraft are calculated in the subsonic regime by use of the doublet lattice method method using NASTRAN software. These forces are further converted into the Laplace domain for aeroservoelasticity studies. In the literature, classical methods [1–3] are used to approximate the unsteady generalized forces from the frequency domain $Q(k)$ to the Laplace domain $Q(s)$. These methods are as follows: least square (LS) [1], matrix Padé [2], and minimum state (MS) [3]. In this paper, we present a new mixed method based on a combination of the most well known methods available in the aeroservoelasticity literature: the LS and MS methods. We found that our method gives very good results with respect to the LS method and combines the advantages of the two classical methods LS and MS. Flutter results are presented for a business aircraft with 44 symmetric modes and 50 antisymmetric modes.

New Method Presentation

This new method uses a combination of the two analytical forms for the conversion of the aerodynamic unsteady forces from the frequency into the Laplace domain given by the LS and MS methods. Thus, the general form of the unsteady aerodynamic forces in the Laplace domain s is written in the following form:

$$Q(s) = A_0 + A_1 s + A_2 s^2 + \sum_{i=1}^{n_{\text{Lags}}} \frac{A_{i+2}^{\text{LS}}}{s + b_i} s + D[\bar{s}I - R]^{-1} E s \qquad (1)$$

where $Q(s)$ are the unsteady aerodynamic forces, b_i are the lag terms, A_0, A_1, A_2, and A_{i+2}^{LS} are $(n * n)$ matrices, D is a $(n * n_{\text{Lags}})$ matrix, R is a $(n_{\text{Lags}} * n_{\text{Lags}})$ diagonal matrix, and E is a $(n_{\text{Lags}} * n)$ matrix. The LS exponent of A_{i+2}^{LS} shows the LS part of Eq. (1), n is the number of modes ($n = 50$ for antisymmetric modes case and $n = 44$ for symmetric modes case), and n_{Lags} is the number of lag terms [$n_{\text{Lags}} = 8$ in the LS method and $n_{\text{Lags}} = 1, 2, 3, 4$ in the mixed state (MxState) method]. The two last terms on the right hand side of Eq. (1) are

$$\sum_{i=1}^{n_{\text{Lags}}} \frac{A_{i+2}^{\text{LS}}}{s + b_i} s$$

and $D[sI - R]^{-1} Es$. The analytical form of the first term is used in the LS method, and the analytical form of the second term is used in the MS method. We can write the second term $D[sI - R]^{-1} Es$ in the form of the first term. Thus, Eq. (1) can be rewritten as follows:

$$Q(s) = A_0 + A_1 s + A_2 s^2 + \sum_{i=1}^{n_{\text{Lags}} = n_{\text{LS}} = n_{\text{MS}}} \frac{A_{i+2}}{s + b_i} s \qquad (2)$$

where n_{LS} and n_{MS} represent the number of total lag terms of the LS and MS methods, respectively. As observed in Eq. (2), for simplicity, we assumed that the lag terms calculated by the LS method are equal to the lag terms calculated by the MS method ($b_{i_{\text{LS}}} = b_{i_{\text{MS}}}$), from which the number of lags are also assumed to be equal ($n_{\text{LS}} = n_{\text{MS}}$), because of their common denominator. This assumption is done to simplify the new method's formulation. Once the values of $A_0, A_1, A_2, A_3, \ldots$ and lag terms b_i (where $i = 1, 2, \ldots, n_{\text{Lags}}$) are calculated by the LS method, we will pass from the standard LS form to the new form defined by Eq. (1), and the transfer function

$$\sum_{i=1}^{n_{\text{Lags}}} \frac{A_{i+2}}{s + b_i}$$

is expressed as

$$\sum_{i=1}^{n_{\text{Lags}}} \frac{A_{i+2}}{s + b_i} = \sum_{i=1}^{n_{\text{LS}}} \frac{A_{i+2}^{\text{LS}}}{s + b_i} + D[sI - R]^{-1} E \qquad (3)$$

In the next section, we will show the calculation of the unknown matrices on the right side of Eq. (3) from the known matrices on the left side of Eq. (3) for the easiest case involving two lags ($n_{\text{LS}} = n_{\text{MS}} = 2$). The A_3 and A_4 matrices [left side of Eq. (3)] may be expressed under the general form

$$A_3 = \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \ldots & a_{1m}^3 \\ a_{21}^3 & a_{22}^3 & a_{23}^3 & \ldots & a_{2m}^3 \\ a_{31}^3 & a_{31}^3 & a_{33}^3 & \ldots & a_{3m}^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}^3 & a_{n2}^3 & a_{n3}^3 & \ldots & a_{nm}^3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} a_{11}^4 & a_{12}^4 & a_{13}^4 & \ldots & a_{1m}^4 \\ a_{21}^4 & a_{22}^4 & a_{23}^4 & \ldots & a_{2m}^4 \\ a_{31}^4 & a_{31}^4 & a_{33}^4 & \ldots & a_{3m}^4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}^4 & a_{n2}^4 & a_{n3}^4 & \ldots & a_{nm}^4 \end{bmatrix} \qquad (4)$$

where the elements of matrices A_3 and A_4 are a_{ij}^3 and a_{ij}^4 (exponents 3

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and 4 represent the indices of the A_3 and A_4 matrices). First, we write A_3 in a form where the coefficients in rows $2, 3, \ldots, n$ are written as a function of the first-row coefficients, as shown in the next equation:

$$A_3 = \begin{bmatrix} a_{11}^3 * (1) & a_{12}^3 * (1) & a_{13}^3 * (1) & \ldots & a_{1m}^3 * (1) \\ a_{11}^3 * (\alpha_2^3) + r_{21}^3 & a_{12}^3 * (\alpha_2^3) + r_{22}^3 & a_{13}^3 * (\alpha_2^3) + r_{23}^3 & \ldots & a_{1m}^3 * (\alpha_2^3) + r_{2m}^3 \\ a_{11}^3 * (\alpha_3^3) + r_{31}^3 & a_{12}^3 * (\alpha_3^3) + r_{32}^3 & a_{13}^3 * (\alpha_3^3) + r_{33}^3 & \ldots & a_{1m}^3 * (\alpha_3^3) + r_{3m}^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{11}^3 * (\alpha_n^3) + r_{n1}^3 & a_{12}^3 * (\alpha_n^3) + r_{n2}^3 & a_{13}^3 * (\alpha_n^3) + r_{n3}^3 & \ldots & a_{1m}^3 * (\alpha_n^3) + r_{nm}^3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ \alpha_2^3 & 0 \\ \alpha_3^3 & 0 \\ \ldots & \ldots \\ \alpha_n^3 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \ldots & a_{1m}^3 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^3 & r_{22}^3 & r_{23}^3 & \ldots & r_{2m}^3 \\ r_{31}^3 & r_{32}^3 & r_{33}^3 & \ldots & r_{3m}^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^3 & r_{n2}^3 & r_{n3}^3 & \ldots & r_{nm}^3 \end{bmatrix} \tag{5}$$

where $\alpha_2^3, \alpha_3^3, \ldots, \alpha_n^3$ are real factors arbitrarily chosen, r_{ij}^3 ($i = 1$ to n and $j = 1$ to m) are real residual values. The A_4 matrix can be written in the same form as the A_3 matrix [as in Eq. (5)], but its form is rearranged as follows:

$$A_4 = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_2^4 \\ 0 & \alpha_3^4 \\ \ldots & \ldots \\ 0 & \alpha_n^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & \ldots & a_{1m}^4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^4 & r_{22}^4 & r_{23}^4 & \ldots & r_{2m}^4 \\ r_{31}^4 & r_{32}^4 & r_{33}^4 & \ldots & r_{3m}^4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^4 & r_{n2}^4 & r_{n3}^4 & \ldots & r_{nm}^4 \end{bmatrix} \tag{6}$$

To write the representative form of the A_{i+2} matrix ($i = 1, 2, \ldots, n_{\mathrm{LS}}$), we use the same form as for the A_3 [see Eq. (5)] or the A_4 matrix [see Eq. (6)], then we change the position of the $[1, \alpha_2^{i+2}, \alpha_3^{i+2}, \ldots, \alpha_n^{i+2}]$ column and the $[a_{11}^{i+2}, a_{12}^{i+2}, a_{13}^{i+2}, \ldots, a_{1m}^{i+2}]$ row to the ith column and ith row positions. Now, by computing the summation

$$\sum_{i=1}^{n_{\mathrm{Lags}}} \frac{A_{i+2}}{s + b_i}$$

(where $n_{\mathrm{Lags}} = 2) = [A_3/(s + b_1)] + [A_4/(s + b_2)]$, we obtain

$$\frac{A_3}{s+b_1} + \frac{A_4}{s+b_2} = \frac{1}{s+b_1} \left\{ \begin{bmatrix} 1 & 0 \\ \alpha_2^3 & 0 \\ \alpha_3^3 & 0 \\ \ldots & \ldots \\ \alpha_n^3 & 0 \end{bmatrix} \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \ldots & a_{1m}^3 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^3 & r_{22}^3 & r_{23}^3 & \ldots & r_{2m}^3 \\ r_{31}^3 & r_{32}^3 & r_{33}^3 & \ldots & r_{3m}^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^3 & r_{n2}^3 & r_{n3}^3 & \ldots & r_{nm}^3 \end{bmatrix} \right\}$$
$$+ \frac{1}{s+b_2} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & \alpha_2^4 \\ 0 & \alpha_3^4 \\ \ldots & \ldots \\ 0 & \alpha_n^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & \ldots & a_{1m}^4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^4 & r_{22}^4 & r_{23}^4 & \ldots & r_{2m}^4 \\ r_{31}^4 & r_{32}^4 & r_{33}^4 & \ldots & r_{3m}^4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^4 & r_{n2}^4 & r_{n3}^4 & \ldots & r_{nm}^4 \end{bmatrix} \right\} \tag{7}$$

Then, Eq. (7) may further be written in the following form:

$$\frac{A_3}{s+b_1} + \frac{A_4}{s+b_2} = \begin{bmatrix} 1 & 1 \\ \alpha_2^3 & \alpha_2^4 \\ \alpha_3^3 & \alpha_3^4 \\ \ldots & \ldots \\ \alpha_n^3 & \alpha_n^4 \end{bmatrix} \begin{bmatrix} 1/(s+b_1) & 0 \\ 0 & 1/(s+b_2) \end{bmatrix} \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \ldots & a_{1m}^3 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & \ldots & a_{1m}^4 \end{bmatrix} + [1/(s+b_1)] \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^3 & r_{22}^3 & r_{23}^3 & \ldots & r_{2m}^3 \\ r_{31}^3 & r_{32}^3 & r_{33}^3 & \ldots & r_{3m}^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^3 & r_{n2}^3 & r_{n3}^3 & \ldots & r_{nm}^3 \end{bmatrix}$$
$$+ [1/(s+b_2)] \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ r_{21}^4 & r_{22}^4 & r_{23}^4 & \ldots & r_{2m}^4 \\ r_{31}^4 & r_{32}^4 & r_{33}^4 & \ldots & r_{3m}^4 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ r_{n1}^4 & r_{n2}^4 & r_{n3}^4 & \ldots & r_{nm}^4 \end{bmatrix} \tag{8}$$

By the identification of terms, we found the following unknown matrix values [see Eq. (3)]:

$$A_3^{\mathrm{LS}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ r_{21}^3 & r_{22}^3 & r_{23}^3 & \dots & r_{2m}^3 \\ r_{31}^3 & r_{32}^3 & r_{33}^3 & \dots & r_{3m}^3 \\ \dots & \dots & \dots & \dots & \dots \\ r_{n1}^3 & r_{n2}^3 & r_{n3}^3 & \dots & r_{nm}^3 \end{bmatrix}, \qquad A_4^{\mathrm{LS}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ r_{21}^4 & r_{22}^4 & r_{23}^4 & \dots & r_{2m}^4 \\ r_{31}^4 & r_{32}^4 & r_{33}^4 & \dots & r_{3m}^4 \\ \dots & \dots & \dots & \dots & \dots \\ r_{n1}^4 & r_{n2}^4 & r_{n3}^4 & \dots & r_{nm}^4 \end{bmatrix} \tag{9}$$

$$D = \begin{bmatrix} 1 & 1 \\ \alpha_2^3 & \alpha_2^4 \\ \alpha_3^3 & \alpha_3^4 \\ \dots & \dots \\ \alpha_n^3 & \alpha_n^4 \end{bmatrix}, \qquad [sI - R]^{-1} = \begin{bmatrix} 1/(s+b_1) & 0 \\ 0 & 1/(s+b_2) \end{bmatrix}, \quad \text{and} \quad E = \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \dots & a_{1m}^3 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & \dots & a_{1m}^4 \end{bmatrix}$$

We can see that the first-row elements of all of the A_{i+2}^{LS} matrices are equal to zero (=0) and that the first-row elements of the D matrix are equal to one (=1). In sum, we observe that the inverse problem requires only a few matrix manipulations, which are very useful when conserving computation time. Programming the inverse problem does not require iterative solutions. We can see that

1) The formulation of the aerodynamic forces in the Laplace domain as described by Eq. (1) is very simple.
2) It is possible to obtain Eq. (8) as it is represented by Eq. (12) by modifying the standard LS equation as

$$Q(s) = A_0 + A_1 s + A_2 s^2 + \sum_{i=1}^{n_{\mathrm{Lags}}} \frac{A_{i+2}}{s+b_i} s \tag{10}$$

where A_{i+2} is a product of two matrices C_{i+2} and ML_{i+2}.

C_{i+2} are a set of constant matrix that have the general form

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ \alpha_2^{i+2} & 0 & \dots & 0 \\ \alpha_3^{i+2} & 0 & \dots & 0 \\ \dots & 0 & \dots & \dots \\ \alpha_n^{i+2} & 0 & \dots & 0 \end{bmatrix}$$

ML_{i+2} are a set of matrix to compute by an iterative process and have the general form

$$\begin{bmatrix} a_{11}^{i+2} & a_{12}^{i+2} & a_{13}^{i+2} & \dots & a_{1m}^{i+2} \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus, the sum of terms $A_{i+2} = C_{i+2}\mathrm{ML}_{i+2}$, where $i = 1$ to n_{Lags} can be written as follows:

$$\begin{aligned} \frac{A}{s+b_1} + \frac{A_4}{s+b_2} + \dots + \frac{A_{k+2}}{s+b_k} = {} & \frac{1}{s+b_1} \begin{bmatrix} 1 & 0 & \dots & 0 \\ \alpha_2^3 & 0 & \dots & 0 \\ \alpha_3^3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha_n^3 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 & \dots & a_{1m}^3 \\ 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & + \frac{1}{s+b_2} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & \alpha_2^4 & \dots & 0 \\ 0 & \alpha_3^4 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \alpha_n^4 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{11}^4 & a_{12}^4 & a_{13}^4 & 0 & a_{1m}^4 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \dots \frac{1}{s+b_k} \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & \alpha_2^{k+2} \\ 0 & 0 & \dots & \alpha_3^{k+2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_n^{k+2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{11}^{k+2} & a_{12}^{k+2} & a_{13}^{k+2} & 0 & a_{1m}^{k+2} \end{bmatrix} \end{aligned} \tag{11}$$

and will be written under the following condensed form

$$\frac{A_3}{s+b_1} + \frac{A_4}{s+b_2} + \dots + \frac{A_{k+2}}{s+b_k} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_2^3 & \alpha_2^4 & \dots & \alpha_2^{k+2} \\ \alpha_3^3 & \alpha_3^4 & \dots & \alpha_3^{k+2} \\ \dots & \dots & \dots & \dots \\ \alpha_n^3 & \alpha_n^4 & \dots & \alpha_n^{k+2} \end{bmatrix} \begin{bmatrix} 1/(s+b_1) & 0 & \dots & 0 \\ 0 & 1/(s+b_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/(s+b_k) \end{bmatrix} \begin{bmatrix} a_{11}^3 & a_{12}^3 & \dots & a_{1m}^3 \\ a_{11}^4 & a_{12}^4 & \dots & a_{1m}^4 \\ \dots & \dots & \dots & \dots \\ a_{11}^{k+2} & a_{12}^{k+2} & \dots & a_{1m}^{k+2} \end{bmatrix} \tag{12}$$

$$D = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_2^3 & \alpha_2^4 & \dots & \alpha_2^{k+2} \\ \alpha_3^3 & \alpha_3^4 & \dots & \alpha_3^{k+2} \\ \dots & \dots & \dots & \dots \\ \alpha_n^3 & \alpha_n^4 & \dots & \alpha_n^{k+2} \end{bmatrix}, \qquad [sI - R]^{-1} = \begin{bmatrix} 1/(s+b_1) & 0 & \dots & 0 \\ 0 & 1/(s+b_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/(s+b_k) \end{bmatrix}, \qquad E = \begin{bmatrix} a_{11}^3 & a_{12}^3 & \dots & a_{1m}^3 \\ a_{11}^4 & a_{12}^4 & \dots & a_{1m}^4 \\ \dots & \dots & \dots & \dots \\ a_{11}^{k+2} & a_{12}^{k+2} & \dots & a_{1m}^{k+2} \end{bmatrix}$$

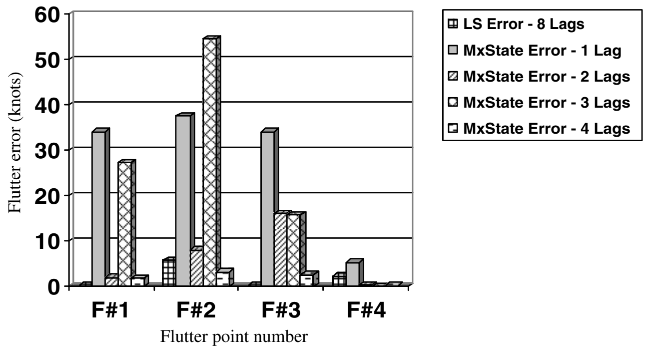

Fig. 1 Flutter speed error results (knots) calculated by the LS method with 8 lags and by the MxState method with 1–4 lags for 44 symmetric modes.

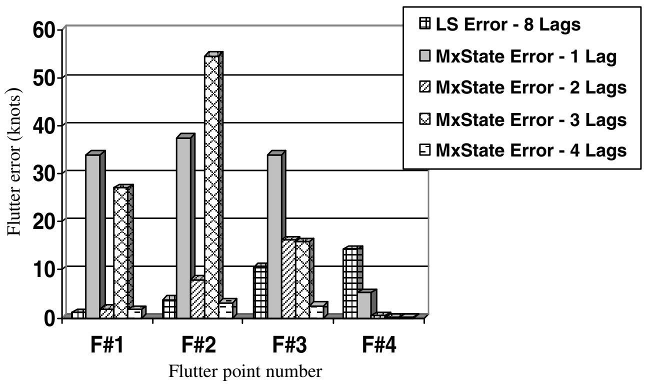

Fig. 2 Flutter speed error results (knots) calculated by the LS method with 8 lags and by the MxState method with 1–4 lags for 50 antisymmetric modes.

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n_{\text{Lags}}$.

By use of Eqs. (10–12), the method of calculation of A_{i+2} coefficients was modified in comparison with the LS method of the same A_{i+2} coefficients calculations. Thus, we shown that it is possible to move from the LS formulation to MS just by replacing

$$\sum_{i=1}^{n_{\text{Lags}}} \frac{A_{i+2}}{s + b_i}$$

term by

$$\sum_{i=1}^{n_{\text{Lags}}} \frac{C_{i+2} * \text{ML}_{i+2}}{s + b_i}$$

and this will allow us to avoid the use of the MS algorithm and use the LS algorithm to get the minimum state formulation.

Results

By use of the approximation of the aerodynamic forces in the *pk* flutter method, we obtained several flutter results. For this business aircraft, 4 flutter values were predicted. Figures 1 and 2 represent, in the form of 3-D bars, the differences in flutter speed values (in knots) calculated with various approximation methods such as LS (with 8 lag terms) and our new MxState method (with 1, 2, 3 and 4 lag terms) implemented in the *pk* flutter method with respect to the flutter speeds calculated with the standard *pk* flutter program. We have called these differences "flutter speed error results." Figures 3 and 4 show, again using 3-D bars, the differences in flutter frequency values (Hz) calculated with various approximation methods such as least square LS (with 8 lag terms) and our new MxState method (with 1, 2, 3, and 4 lag terms) with respect to the flutter frequencies (Hz) calculated by the *pk* flutter standard program.

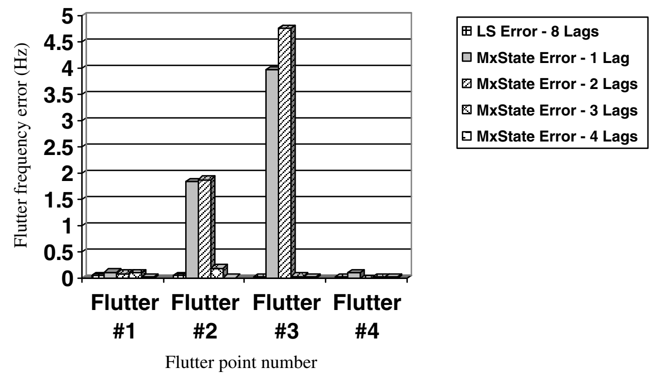

Fig. 3 Flutter frequency error results (Hz) calculated by the LS method with 8 lags and by the MxState method with 1–4 lags for 44 symmetric modes of a business aircraft.

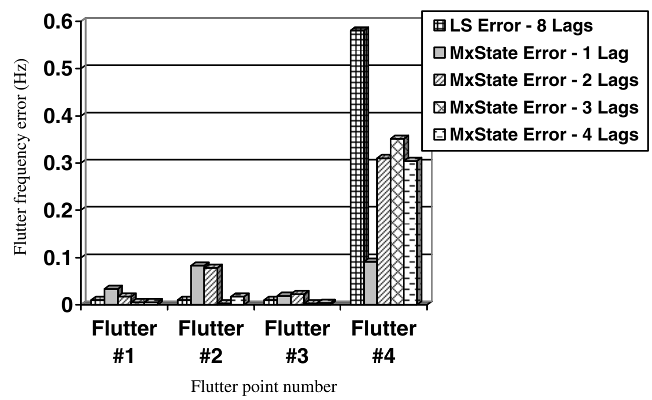

Fig. 4 Flutter frequency error results (Hz) calculated by the LS method with 8 lags and by the MxState method with 1–4 lags for 50 antisymmetric modes of a business aircraft.

Conclusions

This new MxState method will allow us to obtain the MS approximation without passing through a long iterative algorithm. In this manner, we minimize the number of lag terms in the MS approximation term, which means that the lags applied to the two terms:

$$\sum_{i=1}^{n_{\text{Lag}}} \frac{A_{i+2}}{s + b_i} s$$

and $D[sI - R]^{-1} Es$ will be the same. We compared the flutter speeds and frequencies found by the flutter standard nonlinear method with the flutter speeds and frequencies found by the LS approximation method with 8 lag terms, and with the flutter speeds and frequencies found by the new MxState with 1 to 4 lag terms for the CL-604 aircraft.

We found that the MxState method with 4 lags gives results that are very close to the results of the standard *pk* flutter and the LS approximation method with 8 lags. We noticed that the best error values error for both cases (symmetric and antisymmetric modes) are those of the MxState with 4 lags. These results were obtained on a business aircraft with 44 symmetric modes and 50 antisymmetric modes.

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References

[1] Tiffany, S. H., and Adams, W. M., "Nonlinear Programming Extensions to Rational Function Approximation of Unsteady Aerodynamics," NASA TP-2776, July 1988.

[2] Roger, K. L., "Airplane Math Modeling Methods for Active Control Design," *Structural Aspects of Active Controls: Papers Presented at the 44th Meeting of the AGARD Structures and Material Panel, Lisbon, Portugal, 21 April 1977*, AGARD, Neuilly sur Seine, France, 1977, pp. 4.1–4.11; also AGARD CP-228.

[3] Karpel, M., "Design for Flutter Suppression and Gust Alleviation Using State Space Modeling," *Journal of Aircraft*, Vol. 19, No. 3, 1982, pp. 221–227.